Robust Model Reductions for the Boundary Feedback Stabilization of Magnetizable Piezoelectric Beam Equations

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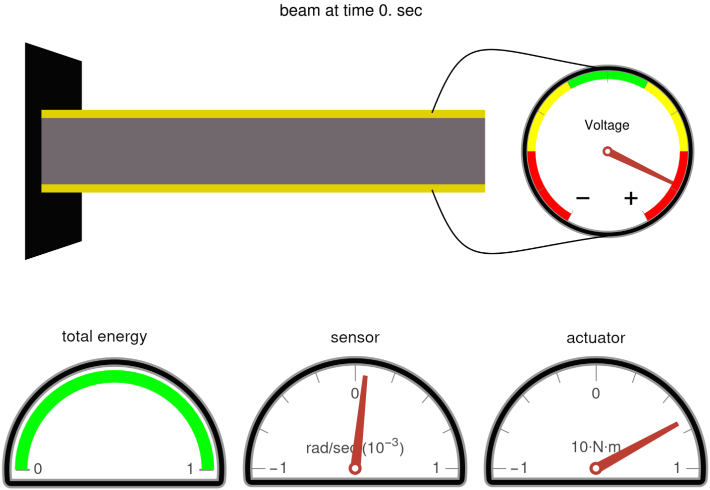
[1]This work is supported by NSF under Cooperative Agreement No. 1849213.

Magnetizable Piezoelectric Beam [2]



[2] [Aydin-Walterman-Leveridge-Özer, Wolfram'23]
https://demonstrations.wolfram.com

Magnetizable Piezoelectric Beam Equations

- $v(x,t)$: Longitudinal displacement of the centerline of the beam
- $p(x,t)$: Total charge accumulated at the electrodes
- $\rho,\ \alpha,\ \beta,\ \gamma, \mu > 0$: Material parameters
- $g(t),\ V(t)$: Boundary controllers

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} = 0, \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0, \quad (x,t) \in (0,L) \times \mathbb{R}^+, \end{cases}$$

$$\begin{cases} v(0,t) = p(0,t) = 0, \\ \alpha v_x(L,t) - \gamma\beta p_x(L,t) = \boxed{-\xi_1 v_t(L,t)}, \\ \beta p_x(L,t) - \gamma\beta v_x(L,t) = \boxed{-\xi_2 p_t(L,t),} \quad t \in \mathbb{R}^+ \end{cases}$$

- $\mu \equiv 0$ (Non-magnetizable or Electrostatic) vs. $\mu \neq 0$ (Magnetizable)

Compact Form of the PDE

Define Matrices

$$\boldsymbol{C}_1 := \begin{bmatrix} \rho & 0 \\ 0 & \mu \end{bmatrix}_{2\times 2}, \quad \boldsymbol{C}_2 := \begin{bmatrix} \alpha & -\gamma\beta \\ -\gamma\beta & \beta \end{bmatrix}_{2\times 2}, \quad \boldsymbol{C}_3 := \begin{bmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{bmatrix}_{2\times 2}.$$

Then the system can be written as

$$\left\{ \boldsymbol{C}_1 \begin{bmatrix} v_{tt} \\ p_{tt} \end{bmatrix} - \boldsymbol{C}_2 \begin{bmatrix} v_{xx} \\ p_{xx} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, x \in (0, L) \right.$$

$$\left\{ \begin{array}{l} v(0,t) = p(0,t) = 0, \\ \boldsymbol{C}_2 \begin{bmatrix} v_x \\ p_x \end{bmatrix} (L,t) = -\boldsymbol{C}_3 \begin{bmatrix} \dot{v}(L) \\ \dot{p}(L) \end{bmatrix}, \quad t \in \mathbb{R}^+ \end{array} \right.$$

$$[v, p, v_t, p_t](x, 0) = [v_0, p_0, v_1, p_1](x), \; x \in [0, L]$$

Known Stability Results for the Collocated System

• [Morris-Özer-SICON'14] Exponential stability with only one state feedback $\xi_1 = 0, \xi_2 \neq 0$ or $\xi_1 \neq 0, \xi_2 = 0$ is not possible in the natural energy space

$$\mathrm{H} = (H^1_*(0,L))^2 \times (L^2(0,L))^2$$

where $H^1_*(0,L) := \{f \in H^1(0,L) : \; f(0) = 0\}$.

• [Özer-MCSS'15] Exponential stability is possible in a more regular space with several number-theoretical constraints on the material parameters.

• [Ramos, et al-ZAMP'18] Exponential stability in H with both state feedback, i.e. $\xi_1 \neq 0, \xi_2 \neq 0$. However, the decay rate is implicit in terms of ξ_1, ξ_2. It is not possible to find the maximal decay rate since the proof is based on an observability result.

An Observability Result with Optimal Observation Time

Theorem (Wilson-Özer-APAN'21)

Consider the weak solutions of the control-free system, i.e. $\xi_1 = \xi_2 = 0$, *in the natural energy space* H. *For any* $T > \frac{2L}{\max(\zeta_1, \zeta_2)}$ *there exists a constant* $C(T) > 0$ *such that*

$$\int_0^T \left(\rho \left|v_t(L,t)\right|^2 + \mu \left|p_t(L,t)\right|^2\right) dt \geq C(T)E(0)$$

for all initial data $[v_0, p_0, v_1, p_1] \in \mathrm{H}$.

WANT: An alternate proof (not by a thorough spectral approach) so that it can be used for Finite-Difference-based model reductions.

The Energy is Dissipative

The natural energy of the solutions is defined by

$$E(t) := \frac{1}{2}\int_0^L \left[\boldsymbol{C}_1 \begin{bmatrix} v_t \\ p_t \end{bmatrix} \cdot \begin{bmatrix} v_t \\ p_t \end{bmatrix} + \boldsymbol{C}_2 \begin{bmatrix} v_x \\ p_x \end{bmatrix} \cdot \begin{bmatrix} v_x \\ p_x \end{bmatrix} \right] dx.$$

Lemma (Özer-Morris-IEEE-CDC'13)

The energy $E(t)$ is dissipative, i.e. for all $t \geq 0$,

$$\dot{E}(t) = -\xi_1 |v_t(L,t)|^2 - \xi_2 |p_t(L,t)|^2 \leq 0.$$

WANT TO PROVE: $E(t)$ decays exponentially, i.e. there exists constants $M, \sigma > 0$ such that

$$E(t) \leq M E(0) e^{-\sigma t}, \qquad \forall t > 0.$$

Alternate Proof by a Lyapunov Approach

For $\delta > 0$, define an energy-like functional $F(t)$ and perturbed energy $E_\delta(t)$

$$F(t) := \int_0^L (\rho v_t x v_x + \mu p_t x p_x)\, dx,$$
$$E_\delta(t) := E(t) + \delta F(t),$$

Lemma

Letting $0 < \delta < \frac{1}{\eta L}$, *for all* $t > 0$, $E_\delta(t)$*is equivalent to* $E(t)$, *i.e.*

$$(1 - \delta\eta L)\, E(t) \le E_\delta(t) \le (1 + \delta\eta L)\, E(t),$$

where $\eta := \max\left\{ \sqrt{\frac{\rho}{\alpha_1}} + \sqrt{\frac{\mu\gamma^2}{\alpha_1}}, \sqrt{\frac{\mu}{\beta}} + \sqrt{\frac{\mu\gamma^2}{\alpha_1}} \right\}.$

Exponential Stability

Theorem (Özer-Emran-Aydin'23-submitted-arXiv:2306.10705)

The energy $E(t)$ of solutions decays exponentially, i.e.

$$E(t) \le M E(0) e^{-\sigma t}, \qquad \forall t > 0,$$
$$\sigma(\delta) = \delta\,(1 - \delta L \eta), \qquad M(\delta) = \frac{1 + \delta L \eta}{1 - \delta L \eta},$$

where for any $\epsilon > 0$

$$\delta(\xi_1, \xi_2, \epsilon) < \frac{1}{L} min \left\{ \frac{1}{\eta}, f_1(\xi_1, \epsilon), f_2(\xi_2, \epsilon) \right\},$$
$$f_1(\xi_1, \epsilon) := \frac{2\xi_1 \alpha_1}{\rho \alpha_1 + (1+\epsilon)\xi_1^2}, \quad f_2(\xi_2, \epsilon) := \frac{2\xi_2 \epsilon \alpha_1 \beta}{\epsilon \mu \alpha_1 \beta + (\epsilon \alpha + \gamma^2 \beta)\xi_2^2}.$$

- Proof by Grönwall's inequality.

Maximal Decay Rate with which ξ_1, ξ_2, ϵ?

Recall

$$E(t) \leq ME(0)e^{-\sigma(\delta)t}, \text{ where } \sigma(\delta) = \delta\left(1 - \delta L\eta\right).$$

GOAL: Find ϵ, ξ_1, ξ_2 that yields $\sigma_{\max}(\delta)$.

$$\sigma_{max}(\delta) = \frac{1}{4\eta L} \text{ is achieved at } \delta(\xi_1, \xi_2, \epsilon) = \frac{1}{2\eta L}.$$

Recall

$$\delta(\xi_1, \xi_2, \epsilon) < \frac{1}{L}\min\left\{\frac{1}{\eta}, f_1(\xi_1, \epsilon), f_2(\xi_2, \epsilon)\right\}.$$

GOAL (cont'd): For which ξ_1, ξ_2, $\delta = \frac{1}{2\eta L}$ is achieved?
It is sufficient to have

$$f_1(\xi_1, \epsilon) > \frac{1}{2\eta}, \text{ and } f_2(\xi_2, \epsilon) > \frac{1}{2\eta}.$$

Maximal Decay Rate - Result

Theorem (Özer-Emran-Aydin'23-submitted-arXiv:2306.10705)

Define non-negative constants

$$
\begin{aligned}
c_1 &:= \frac{2\alpha_1\eta-\sqrt{4\alpha_1^2\eta^2-(1+\epsilon)\rho\alpha_1}}{1+\epsilon},\\
c_2 &:= \frac{2\alpha_1\eta+\sqrt{4\alpha_1^2\eta^2-(1+\epsilon)\rho\alpha_1}}{1+\epsilon},\\
c_3 &:= \frac{2\epsilon\alpha_1\beta\eta-\sqrt{4(\epsilon\alpha_1\beta\eta)^2-(\epsilon\alpha+\beta\gamma^2)\epsilon\mu\alpha_1\beta}}{\epsilon\alpha+\beta\gamma^2},\\
c_4 &:= \frac{2\epsilon\alpha_1\beta\eta+\sqrt{4(\epsilon\alpha_1\beta\eta)^2-(\epsilon\alpha+\beta\gamma^2)\epsilon\mu\alpha_1\beta}}{\epsilon\alpha+\beta\gamma^2},
\end{aligned}
\tag{1}
$$

For any ϵ *satisfying* $\frac{\beta\gamma^2\mu}{4\alpha_1\beta\eta^2-\alpha\mu} < \epsilon < \frac{4\alpha_1\eta^2-\rho}{\rho}$, *the maximal decay rate* $\sigma_{max}(\delta)$ *is achieved for the closed-loop system system as*

$$\xi_1 \in (c_1, c_2), \qquad \xi_2 \in (c_3, c_4).$$

What's up with Realistic Material Parameters?

• Note that the mechanical wave propagation speed is **MUCH LESS** than the electromagnetic one, i.e.

$$\sqrt{\frac{\alpha}{\rho}} << \sqrt{\frac{\beta}{\mu}} \approx \text{Speed of Light}$$

Table: Realistic Piezoelectric Material Parameters

Property	Symbol	Value	Unit
Length of the beam	L	1	m
Mass density	ρ	6000	$\mathrm{kg/m^3}$
Magnetic permeability	μ	10^{-6}	H/m
Elastic stiffness	α	10^{9}	$\mathrm{N/m^2}$
Piezoelectric constant	γ	10^{-3}	$\mathrm{C/m^3}$
Impermittivity	β	10^{12}	m/F

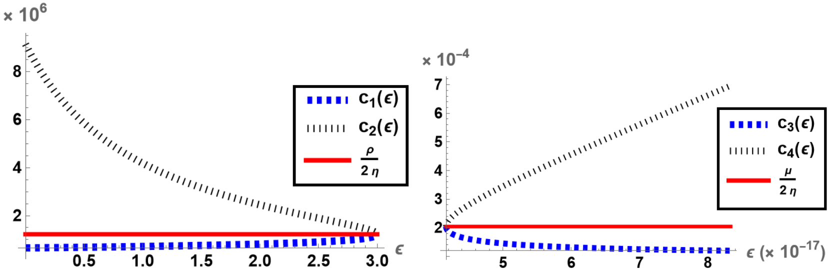


Figure: For realistic material constants, graphs of c_1, c_2, c_3, c_4 with respect to ϵ.

Letting $\epsilon = 1$, and realistic parameters, $\sigma_{\max} = \frac{1}{4\eta L} \approx 102.04$.

$$\begin{aligned} (c_1, c_2) &\approx (7.17 \times 10^5,\ 4.18 \times 10^6), \\ (c_3, c_4) &\approx (1.02 \times 10^{-4},\ 9.78 \times 10^9). \end{aligned} \tag{2}$$

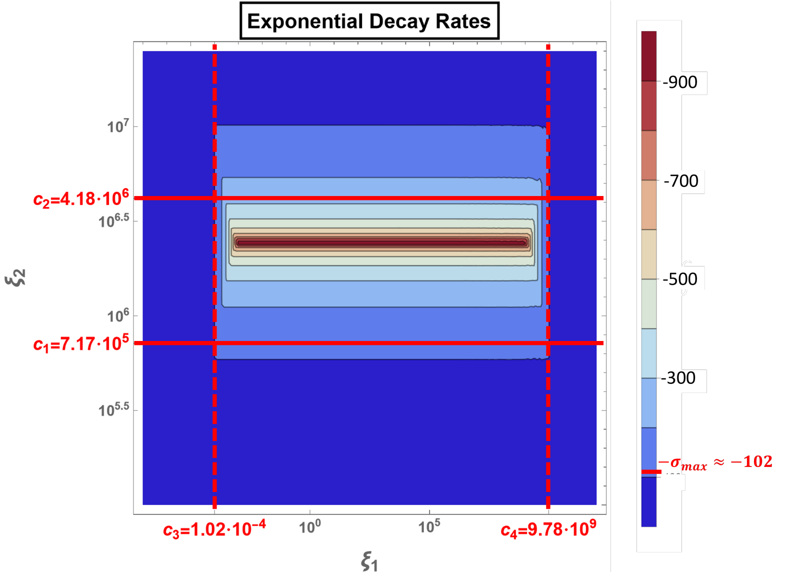


Figure: Decay rates with respect to feedback amplifiers ξ_1 and ξ_2. Red lines delineate the intervals (c_1, c_2) and (c_3, c_4).

Decay Rates

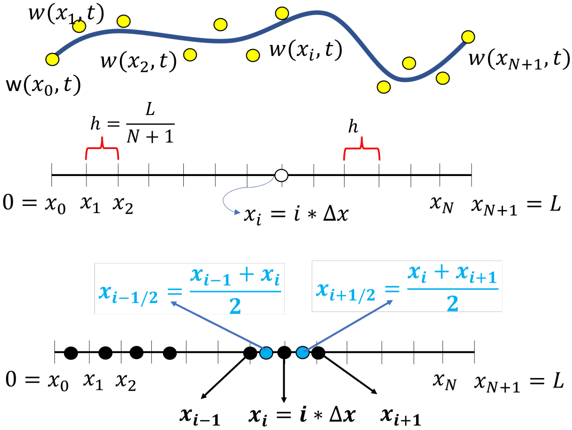
ξ_1 \ ξ_2	10^5	$10^{5.5}$	10^6	$10^{6.5}$	10^7	$10^{7.5}$
10^{-7}	-0.1	-0.1	-0.1	-0.1	-0.1	-0.1
10^{-5}	-10	-10	-10	-10	-10	-10
10^{-3}	-17	-53	-177	-421	-101.9	-32
10^0	-17	-53	-177	-421	-101.9	-32
10^3	-17	-53	-177	-421	-101.9	-32
10^6	-17	-53	-177	-421	-101.9	-31
10^9	-17	-53	-177	-421	-101	-30
10^{10}	-17	-52	-100	-100	-97	-23
10^{11}	-10	-10	-10	-10	-10	-9

Table: The $\max\{\Re(\mu_k)\}$ values for different choices of feedback amplifiers ξ_1 and ξ_2. Red boxes indicate the intervals (c_1, c_2) and (c_3, c_4), where the maximal decay rate is achieved.

Semi-Discretization in Space Variable x: FD vs. ORFD

Standard Finite Difference Discretization

Order-reduced Finite Difference Discretization



ORFD in a Nutshell!

Consider the average over the control volume $\left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$,

$$\frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} v_t(x,t)dx \approx \frac{1}{2}\left[v(x_{i+\frac{1}{2}},t) + v(x_{i-\frac{1}{2}},t)\right] \text{ trapezoidal rule}$$

$$\approx \frac{1}{4}\left(v_{i+1} + 2v_i + v_{i-1}\right)$$

$$\frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} v_{xx}(x,t)dx \approx \frac{1}{h}\left[v_{i+1} - 2v_i + v_{i-1}\right]. \qquad \text{central difference}$$

Discretization Matrices

Define the matrices

$$\boldsymbol{A}_h := \frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & \dots & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & -1 & 2 & -1 & 0 \\ 0 & \dots & \dots & 0 & -1 & 2 & -1 \\ 0 & \dots & \dots & \dots & 0 & -1 & 1 \end{bmatrix}_{N+1\times N+1},$$

$$\boldsymbol{B}_h := \frac{1}{h}\begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix}_{N+1\times N+1}.$$

Mass Matrices

$$\boldsymbol{M}_{FEM} := \frac{1}{6}\begin{bmatrix} 4 & 1 & 0 & \dots & \dots & \dots & 0 \\ 1 & 4 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 4 & 1 & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 & 4 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 & 4 & 1 \\ 0 & \dots & \dots & \dots & 0 & 1 & 2 \end{bmatrix}_{N+1 \times N+1},$$

$$\boldsymbol{M}_{ORFD} := \frac{1}{4}\begin{bmatrix} 2 & 1 & 0 & \dots & \dots & \dots & 0 \\ 1 & 2 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 2 & 1 & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 & 2 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 & 2 & 1 \\ 0 & \dots & \dots & \dots & 0 & 1 & 1 \end{bmatrix}_{N+1 \times N+1}.$$

Model Reductions

The reduced system is

$$(\boldsymbol{C}_1 \otimes \boldsymbol{M}) \begin{bmatrix} \ddot{\vec{v}} \\ \ddot{\vec{p}} \end{bmatrix} + (\boldsymbol{C}_2 \otimes \boldsymbol{A}_h) \begin{bmatrix} \vec{v} \\ \vec{p} \end{bmatrix} + (\boldsymbol{C}_3 \otimes \boldsymbol{B}_h) \begin{bmatrix} \dot{\vec{v}} \\ \dot{\vec{p}} \end{bmatrix} = 0.$$

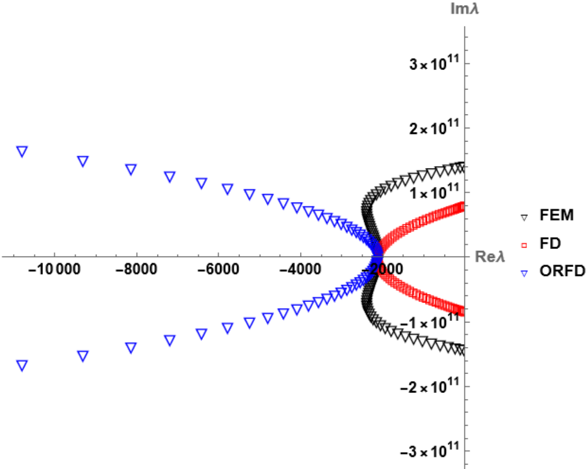
where $\otimes$ is Kronecker product.

Different discretization methods can be represented by changing $\boldsymbol{M}$ such as

- Standard Finite Differences: $\boldsymbol{M} = \boldsymbol{M}_{FD} = \boldsymbol{I}$: Identity Matrix
- Finite Element Method: $\boldsymbol{M} = \boldsymbol{M}_{FEM}$
- Order-Reduced Finite Differences: $\boldsymbol{M} = \boldsymbol{M}_{ORFD}$

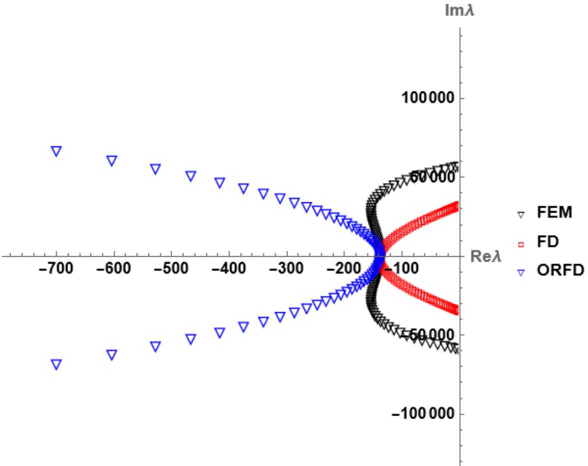
Eigenvalues FD vs. FEM vs. ORFD (First Branch)

- FD and FEM: Lacks of exp. stability unifomly as $h \to 0$ since $\mathrm{Re}\lambda \to 0$.
- ORFD: Preserves exp. stability uniformly as $h \to 0$ since $\max \mathrm{Re}\lambda << 0$

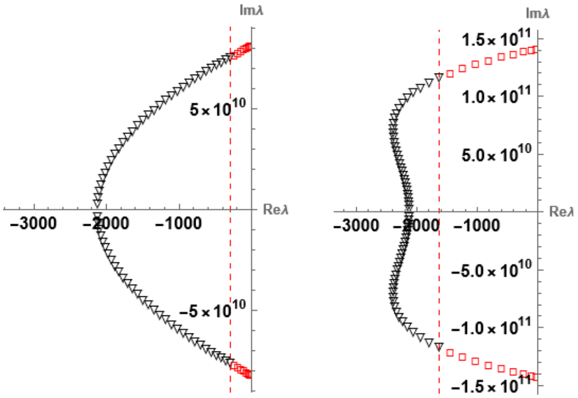


Eigenvalues FD vs. FEM vs. ORFD (Second Branch)

- FD and FEM: Lacks of exp. stability unifomly as $h \to 0$ since $\mathrm{Re}\lambda \to 0$.
- ORFD: Preserves exp. stability uniformly as $h \to 0$ since $\max \mathrm{Re}\lambda << 0$.



- [3] Filter out high-frequency eigenvalues so that $\max \mathrm{Re}\lambda < 0$.



[3] A.Ö. Ozer, R. Emran (2023), *The Direct Fourier Filtering Technique for the Boundary Damped Magnetizable Piezoelectric Beams*, preprint.

Exponential Stability of ORFD: Uniformly as $h \to 0$

Theorem (Özer-Aydin-Walterman'23 Submitted-arXiv:2309.07492)

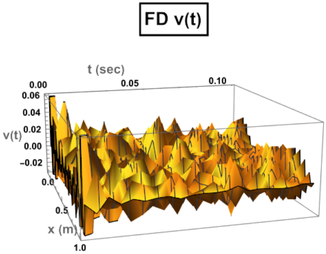
For any $\xi_1, \xi_2 > 0$, *and*

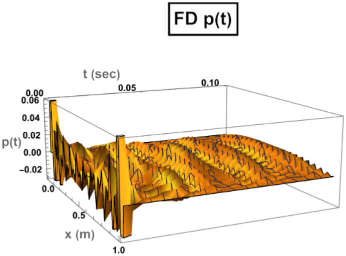
$$\delta < \frac{1}{L} \min \left(\frac{1}{\eta}, \frac{2\xi_1 \alpha_1}{\alpha_1 \rho + \xi_1^2}, \frac{4\xi_2 \beta \alpha_1}{2\alpha_1 \beta \mu + (\alpha + \gamma^2 \beta)\xi_2^2} \right),$$

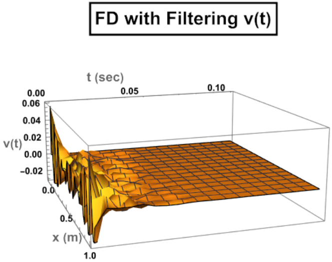
the energy $E_h^{\mathrm{ORFD}}(t)$ *decays exponentially, i.e.*

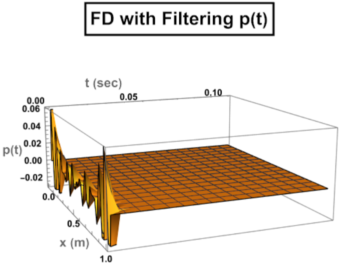
$$\frac{dE_h^{\mathrm{ORFD}}(t)}{dt} \leq \frac{1 + \delta L \eta}{1 - \delta L \eta} e^{-\delta(1 - \delta L \eta)t} E_h^{\mathrm{ORFD}}(0).$$

Simulations with High-Frequency Initial Data (FD)



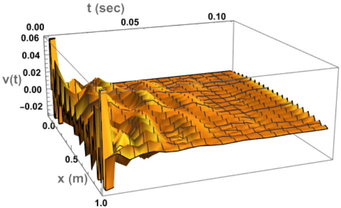




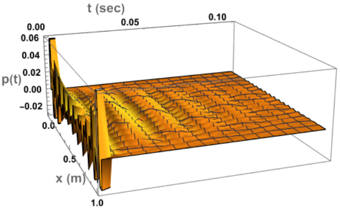


Simulations with High-Frequency Initial Data (FEM)

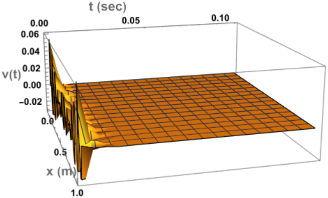
FEM v(t)



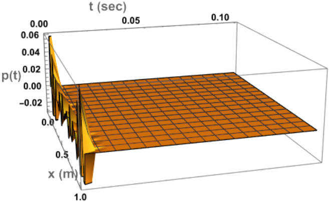
FEM p(t)



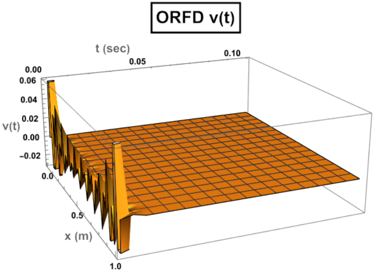
FEM with Filtering v(t)

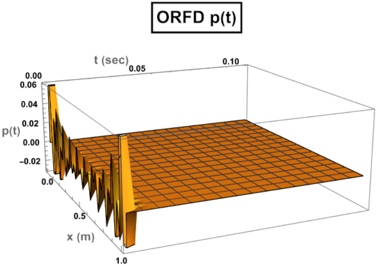


FEM with Filtering p(t)



Simulations with High-Frequency Initial Data (ORFD)





- No need of a numerical filtering!

Generalization to 2D

Consider

$$\begin{cases} \ddot{u} - \Delta u = 0, & \Omega \times \mathbb{R}^+, \\ u = 0, & \partial\Omega \times \mathbb{R}^+, \\ u(x,y,0) = u^0(x,y), \quad \dot{u}(x,y,0) = u^1(x,y), & \Omega. \end{cases} \tag{3}$$

with the discretization

$$(\boldsymbol{M}_x \otimes \boldsymbol{M}_y)\ddot{\vec{u}} + (\boldsymbol{A}_x \oplus \boldsymbol{A}_y)\vec{u} = 0, \tag{4}$$

Eigenvalues in 2D Wave

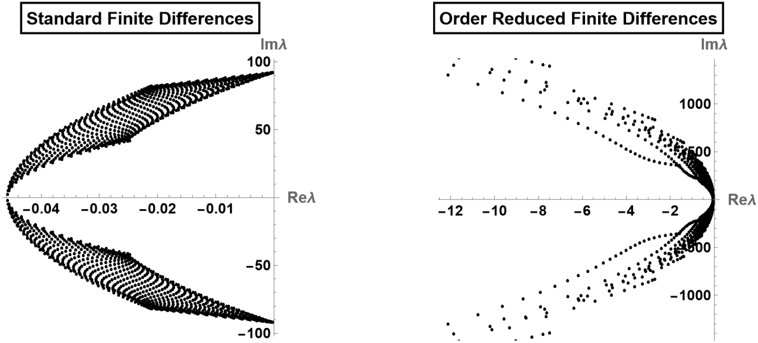


Figure: Eigenvalues of the control problem in 2D where number of the nodes $N = 40 \times 20$, $\xi_x = 0.025$, and $\xi_y = 0.02$.

Eigenvalues in 2D Wave cont'd

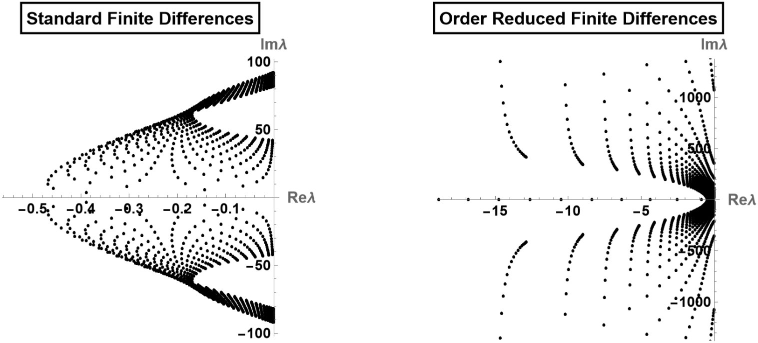


Figure: Eigenvalues of the control problem in 2D where number of the nodes $N = 40 \times 20$, $\xi_x = 2.25$, and $\xi_y = 0$.

Eigenvalues in 2D Wave

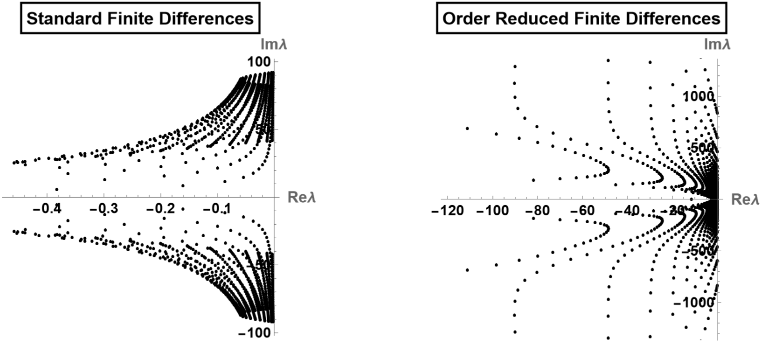


Figure: Eigenvalues of the control problem in 2D where number of the nodes $N = 40 \times 20$, $\xi_x = 0$, and $\xi_y = 1$.

Summary

Discretization methods that retain the control theoric properties:

- Order-Reduced Finite Differences (ORFD)
- Mixed Finite Element Method (MFEM)

Discretization methods that fails to retain the control theoric properties:

- Standard Finite Differences (FD)
- Finite Element Method (FEM)

Remedies:

- Direct Fourier Filtering
- Indirect Filtering

Conclusions and Ongoing Research

- Maximal decay rate analysis - ongoing.
- The overall analysis here aligns with the multi-layer beam designs
  - Mead-Marcus type: [**Aydin**,Özer-IEEE-CDC'22],[**Aydin**,Özer,Walterman-IEEE-LCSS'23], [Özer-**Aydin**-LCSS'23,ESAIM'23]
  - Rao-Nakra type: [Haider-Özer-in progress].
  - Serially-connected designs: [Özer,Nicaise,Akil,Regnier'23-arXiv:2303.05882].
- This type of approach is already being used for the exponential stability of heat-piezoelectric-type transmission line problem in [Özer-Khal.-**Uthman**'23-arXiv:2311.05306].
- Generalization to higher dimensions.