Avoiding Sensor Data Filtering by a Novel Model Reduction for the PDE System of a Multi-Layer Sandwich Beam

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SIAM PNW Conference - WSU, Vancouver, WA

May 21, 2022

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This project is sponsored via NSF under Cooperative Agreement No. 1849213.

Outline

1. PDE
   - Sensor Design (Observability)
2. Finite Difference Model Reduction
   - Order-Reduced Approximated Model
   - Approximation of the Boundary Equation
   - Model Reduction of the Second Equation
3. Discretized Energy is Conserved
4. Uniform Observability as $h \to 0$
5. Ongoing & Open Problems

Cantilever (Clamped-Free) Beam Examples

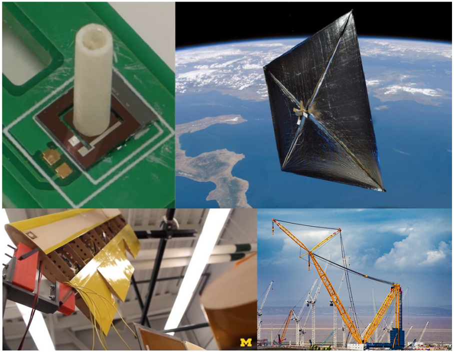

Figure: (Top Left) Microelectromechanical energy harvester [3], (Tom Right) Solar sail "Sunjammer"[4], (Bottom Left) Energy harvesting wing design[5], (Bottom Right) Largest crane aka "Big Carl".

[3] Y.J. Lee et al. (2019) Vortex-induced vibration wind energy harvesting by piezoelectric MEMS device in formation

[4] https://www.nasa.gov

[5] University of Michigan Engineering

Multi-layer Structure

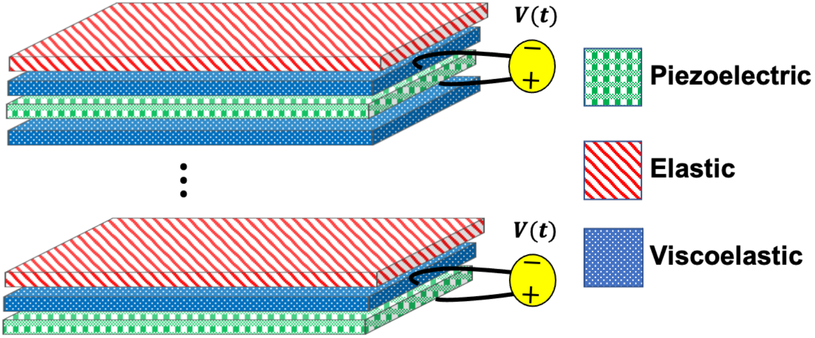

Figure: Multi-layer beam structure with alternating $m+1$ elastic/piezoelectric and m viscoelastic layers.

Three-Layer Beam Model

Let dots and primes denote the differential operators $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$, respectively.

$$\begin{cases} \ddot{z} + z'''' - B\phi' = 0, & \\ \boxed{-C\phi'' + P\phi = -Bz'''}, & (x,t) \in (0,L) \times \mathbb{R}^+ \\ z, z', \phi|_{x=0} = 0, \quad z'', \phi'|_{x=L} = 0, & \\ z'''(L,t) - B\phi(L,t) = 0, & t \in \mathbb{R}^+ \\ z(x,0) = z^0(x), \quad \dot{z}(x,0) = z^1(x), & x \in (0,L) \end{cases} \tag{1}$$

Shear angle ϕ can be expressed in terms of bending z,

$$\phi = -B(-CD_x^2 + P)^{-1} z'''.$$

Compact Form of the Model

Lemma

Define $J := -\frac{1}{C} + \frac{P}{C}(-CD_x^2 + PI)^{-1}$. *Then,* J *is continuous, self-adjoint, and non-positive on* $L^2(0, L)$. *Moreover, for all* $u \in Dom(D_x^2) = \{u \in H^2(0, L) | u(0) = 0, u'(L) = 0\}$

$$J = (-CD_x^2 + PI)^{-1} D_x^2.$$

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$$J = (-CD_x^2 + PI)^{-1} D_x^2.$$

Now, (1) can be rewritten as

$$\begin{cases} \ddot{z} + z'''' + B^2 \boxed{(Jz')'} = 0, & (x,t) \in (0, L) \times \mathbb{R}^+ \\ z, z', \phi|_{x=0} = 0, \quad z'', \phi'|_{x=L} = 0, & \\ z'''(L,t) - B\phi(L,t) = 0, & t \in \mathbb{R}^+ \\ z(x,0) = z^0(x), \quad \dot{z}(x,0) = z^1(x), & x \in (0, L) \end{cases} \tag{2}$$

Energy and Observer

The energy, $E(t)$, of (2) is defined as

$$E(t) = \|(z, \dot{z})\|_E^2 = \frac{1}{2}\int_0^L \left[|\dot{z}|^2 + |z''|^2 - B^2(Jz')z'\right] dx.$$

Note that (2) is a conservative system *i.e.* $\frac{\partial E(t)}{\partial t} = 0,\ \forall t \geq 0.$

GOAL: Prove that for each (z^0, z^1), there exists a constant $C(T) > 0$ such that for $T > T^*$, the so-called exact observability inequality holds

$$\boxed{\int_0^T |\dot{z}(L,t)|^2 dt \geq C(T)E(0) = C(T)E(z^0, z^1).}$$

How to Prove?

Widely-used methods used to prove (7) are:

- Multipliers Method (i.e. see Komornik'97, Masson)

$\rightarrow$ Spectral Analysis (Non-harmonic Fourier series) (i.e. see Komornik & Loreti'05, Springer)

- Microlocal analysis, Carleman Estimates, Moment Problem, etc.

The Exact Observability Result

Theorem

Let $T > 2C$ *with* $C = \max\left\{L, \frac{L^3}{\pi^2}\right\}$. *Then, the following observability inequality holds*

$$\int_0^T |\dot{z}(L,t)|^2 dt \geq \frac{2(T-2C)}{L} E(0).$$

Sketch of the Proof.

The proof is quite technical. The multipliers technique (with the multiplier xz') together with several (Sobolev) functional inequalities are adopted. $\square$

The Exact Observability Result

Theorem

Let $T > 2C$ *with* $C = \max\left\{L, \frac{L^3}{\pi^2}\right\}$. *Then, the following observability inequality holds*

$$\int_0^T |\dot{z}(L,t)|^2 dt \geq \frac{2(T-2C)}{L} E(0).$$

- The observation time $2C$, indeed, is not optimal. This is one of the discrepancies of the multipliers method. Can be improved!

Finite Difference Model Reduction

Let $N \in \mathbb{N}$ be given, and define the mesh parameter $h := \frac{L}{N+1}$. Consider the following uniform discretization of the interval $[0, L]$:

$$0 = x_0 < x_1 < ... < x_j = jh < ... < x_{N-1} < x_N < x_{N+1} = L.$$

Let $z_j = z_j(t) \approx z(x_j, t)$, and $\vec{z} = [z_1, z_2, ..., z_N]^T$.

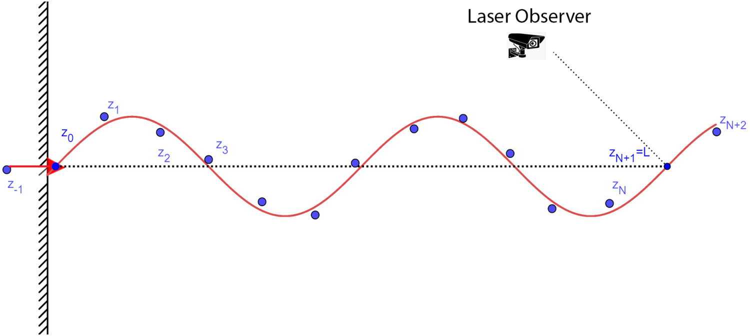


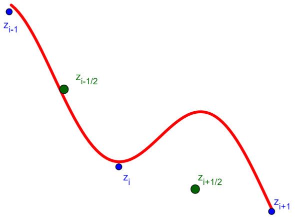
Figure: Space discretization of the $z(x, t)$

Defining the Average Operator - Liu & Guo'20

Define the following average operator

$$z_{i+\frac{1}{2}} = \frac{z_{i+1} + z_i}{2}$$

corresponding to the "extra" points $x_{i+\frac{1}{2}} = \left(i + \frac{1}{2}\right) h$ in between the nodes x_i and x_{i+1} of the standard discretization.



Consider the Taylor Series expansion for a function $f(x)$,

$$\begin{aligned} f(x_{i+1}) &= f(x_{i+\frac{1}{2}}) + \frac{h}{2} f'(x_{i+\frac{1}{2}}) + \frac{h^2}{8} f''((x_{i+\frac{1}{2}}) + \mathcal{O}(h^3) \\ f(x_i) &= f(x_{i+\frac{1}{2}}) - \frac{h}{2} f'(x_{i+\frac{1}{2}}) + \frac{h^2}{8} f''((x_{i+\frac{1}{2}}) + \mathcal{O}(h^3). \end{aligned}$$

Defining the Difference Operators - Liu & Guo'20

Consider the Taylor Series expansion for a function $f(x)$,

$$
\begin{aligned}
f(x_{i+1}) &= f(x_{i+\frac{1}{2}}) + \frac{h}{2} f'(x_{i+\frac{1}{2}}) + \frac{h^2}{8} f''((x_{i+\frac{1}{2}}) + \mathcal{O}(h^3) \\
f(x_i) &= f(x_{i+\frac{1}{2}}) - \frac{h}{2} f'(x_{i+\frac{1}{2}}) + \frac{h^2}{8} f''((x_{i+\frac{1}{2}}) + \mathcal{O}(h^3).
\end{aligned}
$$

Hence

$$
f'(x_{i+\frac{1}{2}}) = \frac{f(x_{i+1}) - f(x_i)}{h} + \mathcal{O}(h^3).
$$

Defining the Difference Operators (cont'd.)

Defining the following difference operators

$$\delta_x z_{i+\frac{1}{2}} = \frac{z_{i+1} - z_i}{h}$$

$$\delta_x^2 z_i = \frac{\delta_x z_{i+\frac{1}{2}} - \delta_x z_{i-\frac{1}{2}}}{h} = \frac{z_{i+1} - 2z_i + z_{i-1}}{h^2}$$

$$\delta_x^3 z_{i+\frac{1}{2}} = \frac{\delta_x^2 z_{i+1} - \delta_x^2 z_i}{h} = \frac{z_{i+2} - 3z_{i+1} + 3z_i - z_{i-1}}{h^3}$$

$$\delta_x^4 z_i = \frac{\delta_x^3 z_{i+\frac{1}{2}} - \delta_x^3 z_{i-\frac{1}{2}}}{h} = \frac{z_{i+2} - 4z_{i+1} + 6z_i - 4z_{i-1} + z_{i-2}}{h^4}$$

Reduction of the PDE

Choosing the states

$$u(x,t) := \dot{z}(x,t), \quad v(x,t) := z'''(x,t),$$

the reduction of the first equation of (2) is

$$\ddot{z} + z'''' - B\phi' = 0 \;\rightarrow\; \dot{u} + v' - B\phi' = 0 \;\rightarrow\; \dot{u}_{i+\frac{1}{2}} + \delta_x v_{i+\frac{1}{2}} - B\delta_x \phi_{i+\frac{1}{2}} = 0.$$

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Substituting $\frac{v_{i+1}-v_i}{2} = v_{i+1} - v_{i+\frac{1}{2}} = v_{i+\frac{1}{2}} - v_i$ yields,

$$v_i = v_{i-\frac{1}{2}} - \frac{h}{2}\dot{u}_{i-\frac{1}{2}} + \frac{Bh}{2}\delta_x \phi_{i-\frac{1}{2}}$$
$$v_i = v_{i+\frac{1}{2}} + \frac{h}{2}\dot{u}_{i+\frac{1}{2}} - \frac{Bh}{2}\delta_x \phi_{i+\frac{1}{2}}$$

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$$v_i = v_{i+\frac{1}{2}} + \frac{h}{2}\dot{u}_{i+\frac{1}{2}} - \frac{Bh}{2}\delta_x \phi_{i+\frac{1}{2}}$$

Hence

$$v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}} + \frac{h}{2}\left(\dot{u}_{i+\frac{1}{2}} + \dot{u}_{i-\frac{1}{2}}\right) - \frac{Bh}{2}\left(\delta_x \phi_{i+\frac{1}{2}} + \delta_x \phi_{i-\frac{1}{2}}\right) = 0.$$

Reduction of the PDE

Choosing the states

$$u(x,t) := \dot{z}(x,t), \quad v(x,t) := z'''(x,t),$$

the reduction of the first equation of (2) is

$$\ddot{z} + z'''' - B\phi' = 0 \;\rightarrow\; \dot{u} + v' - B\phi' = 0 \;\rightarrow\; \dot{u}_{i+\frac{1}{2}} + \delta_x v_{i+\frac{1}{2}} - B\delta_x\phi_{i+\frac{1}{2}} = 0.$$

Substituting $\frac{v_{i+1}-v_i}{2} = v_{i+1} - v_{i+\frac{1}{2}} = v_{i+\frac{1}{2}} - v_i$ yields,

$$v_i = v_{i-\frac{1}{2}} - \frac{h}{2}\dot{u}_{i-\frac{1}{2}} + \frac{Bh}{2}\delta_x\phi_{i-\frac{1}{2}}$$

$$v_i = v_{i+\frac{1}{2}} + \frac{h}{2}\dot{u}_{i+\frac{1}{2}} - \frac{Bh}{2}\delta_x\phi_{i+\frac{1}{2}}$$

Hence

$$v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}} + \frac{h}{2}\left(\dot{u}_{i+\frac{1}{2}} + \dot{u}_{i-\frac{1}{2}}\right) - \frac{Bh}{2}\left(\delta_x\phi_{i+\frac{1}{2}} + \delta_x\phi_{i-\frac{1}{2}}\right) = 0.$$

Substituting $\delta_x^3 z_{i+\frac{1}{2}} = v_{i+\frac{1}{2}}$

$$\boxed{\delta_x^4 z_i + \frac{1}{4}\left(\ddot{z}_{i+1} + 2z_i + \ddot{z}_{i-1}\right) - \frac{B}{2h}\left(\phi_{i+1} - \phi_{i-1}\right) = 0.}$$

What Happens to Boundary Equation?

Consider the discretization of the higher-order boundary equation

$$z'''(L,t) - B\phi(L,t) = 0 \;\rightarrow\; v(L,t) - B\phi(L,t) = 0 \;\rightarrow\; v_{N+1} - B\phi_{N+1} = 0.$$

Substitute v_{N+1} to get

$$v_{N+\frac{1}{2}} - \frac{h}{2}\dot{u}_{N+\frac{1}{2}} + \frac{Bh}{2}\delta_x\phi_{N+\frac{1}{2}} - B\phi_{N+1} = 0.$$

Hence,

$$\boxed{-\delta_x^3 z_{N+\frac{1}{2}} + \frac{h}{4}\left(\ddot{z}_{N+1} + \ddot{z}_N\right) + B\phi_N = 0.}$$

Reduced Model

Finally, the reduced model for (1) is

$$
\begin{cases}
\delta_x^4 z_i + \frac{1}{4}\left(\ddot{z}_{i+1} + 2z_i + \ddot{z}_{i-1}\right) - \frac{B}{2h}\left(\phi_{i+1} - \phi_{i-1}\right) = 0, & i = 1, 2, ..., N, \\
-C\left(\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h}\right) + \frac{Ph}{4}(\phi_{i+1} + 2\phi_i + \phi_{i-1}) & \\
\qquad + \frac{B}{2}\left(\frac{z_{i+2} - 2z_{i+1} + 2z_{i-1} - z_{i-2}}{h^2}\right) = 0, & i = 1, 2, ..., N, \\
-\delta_x^3 z_{N+\frac{1}{2}} + \frac{h}{4}\left(\ddot{z}_{N+1} + \ddot{z}_N\right) + B\phi_N = 0, & \\
\phi_0 = 0 \quad \phi_{N+1} = \phi_N \quad z_0 = z_{-1} = 0 \quad z_{N+2} = 2z_{N+1} - z_N, & \\
\phi_j(0) = \phi^0(x_j) \quad z_j(0) = z^0(x_j) \quad \dot{z}_j(0) = z^1(x_j), & j = 1, 2, ..., N+1.
\end{cases}
$$

Note that the the averaging is the same as the one once the Mixed-Finite Element Method is appropriately implemented.

Shear-Bending Relation

Let $y_j = z_{j+1} - z_{j-1}$,

$$-C\left(\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h}\right) + \frac{Ph}{4}(\phi_{i+1} + 2\phi_i + \phi_{i-1}) + \frac{B}{2}\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) = 0.$$

Hence,

$$\boxed{C\boldsymbol{A}_h\vec{\phi} + \frac{Ph}{4}\boldsymbol{M}\vec{\phi} - \frac{B}{2h}\boldsymbol{A}_h\vec{y} = 0}$$

where $\vec{\phi} = [\phi_1, ..., \phi_N]^T$, $\vec{y} = [y_1, ..., y_N]^T$.

Special Matrices A_h, and M

$$
\boldsymbol{A}_h = \frac{1}{h}\begin{bmatrix}
2 & -1 & 0 & \dots & \dots & \dots & 0 \\
-1 & 2 & -1 & 0 & \dots & \dots & 0 \\
0 & -1 & 2 & -1 & 0 & \dots & 0 \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \\
0 & \dots & 0 & 1 & -2 & 1 & 0 \\
0 & \dots & \dots & 0 & -1 & 2 & -1 \\
0 & \dots & \dots & \dots & 0 & -1 & 1
\end{bmatrix}
$$

$$
\boldsymbol{M} = \begin{bmatrix}
2 & 1 & 0 & \dots & \dots & \dots & 0 \\
1 & 2 & 1 & 0 & \dots & \dots & 0 \\
0 & 1 & 2 & 1 & 0 & \dots & 0 \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \\
0 & \dots & 0 & 1 & 2 & 1 & 0 \\
0 & \dots & \dots & 0 & 1 & 2 & 1 \\
0 & \dots & \dots & \dots & 0 & 1 & 3
\end{bmatrix}
$$

Discretization of the J operator

Lemma

Define $\boldsymbol{J}_h = \left(\boldsymbol{C}\boldsymbol{A}_h + \frac{Ph}{4}\boldsymbol{M}\right)^{-1}\boldsymbol{A}_h$. *Then,* $\boldsymbol{J}_h$ *is a self-adjoint operator. Moreover, for all* $u \in Dom(\boldsymbol{J}_h)$,

$$\begin{aligned}
\boldsymbol{J}_h &= \frac{1}{C}\boldsymbol{I} - \frac{Ph}{4C}\left(C\boldsymbol{A}_h + \frac{Ph}{4}\boldsymbol{M}\right)^{-1}\boldsymbol{M} \\
&= \frac{1}{C}\left(I - \left(\frac{4C}{Ph}\boldsymbol{M}^{-1}\boldsymbol{A}_h + I\right)^{-1}\right)
\end{aligned}$$

Thus,

$$\vec{\phi} = \frac{B}{2h}\boldsymbol{J}_h\vec{y}.$$

Lemma

The matrices $\boldsymbol{M}$ and $\boldsymbol{A}_h$ are positive-definite, symmetric, and simultaneously diagonalizable matrices.

Proof.

The eigenvalues and eigenvectors of matrix $\boldsymbol{A}_h$ are

$$\lambda_k = \frac{4}{h}\sin^2\left(\frac{(2k-1)\pi}{4N+2}\right),$$
$$\varphi^k = \begin{bmatrix}\varphi_{k,1} & \cdots & \varphi_{k,N}\end{bmatrix}^T,$$
$$\varphi_{k,j} = \sin\left(\frac{(2j-1)k\pi}{2N+1}\right), \quad k,j = 1,2,\ldots,N.$$

Moreover, the matrices M and A_h satisfy

$$h\boldsymbol{A_h} = 4I - \boldsymbol{M}.$$

□

Energy of the Approximated Model

Define the discrete energy of the system as

$$E_h(t) := \frac{h}{2}\sum_{j=0}^{N}\left(\dot{z}_{j+\frac{1}{2}}\right)^2 + \frac{h}{2}\sum_{j=0}^{N}\left(\delta_x^2 z_j\right)^2 + \boxed{\frac{B}{4}\sum_{j=0}^{N}\phi_j(z_{j+1}-z_{j-1}).}$$

Lemma

The energy of the system is conservative, i.e.,

$$\frac{dE_h(t)}{dt} = 0.$$

Hence, $E_h(t) = E_h(0)$ *for all* $t \geq 0$.

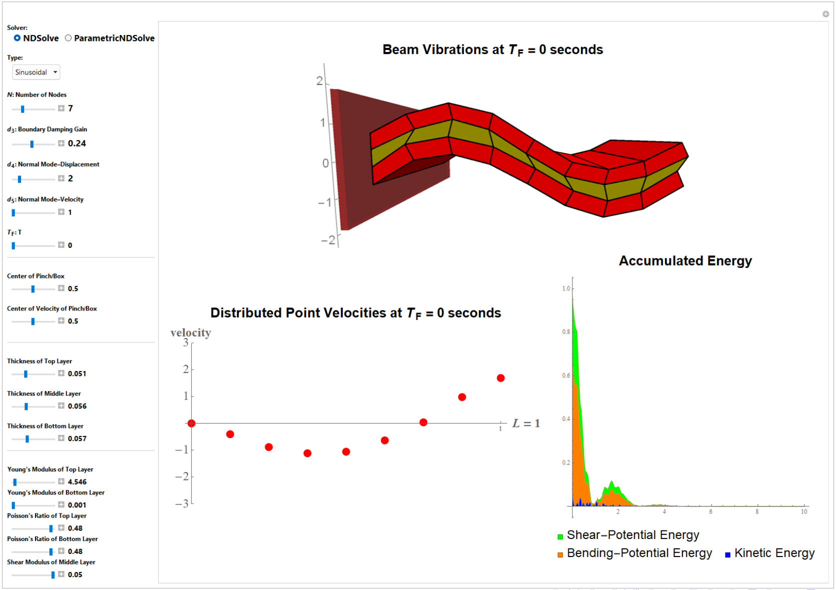
Uniform Observability as $h \to 0$

Theorem (Preprint)

Let $T > 2C$, *where* $C = \max\{1, L\}$. *Then the following observability inequality holds,*

$$E_h(0) \leq \frac{L}{2(T - 2C)} \int_0^T |\dot{z}_{N+1}|^2 dt.$$

Wolfram Demonstration Project - Designing a Stabilizing Controller



Ongoing Problems

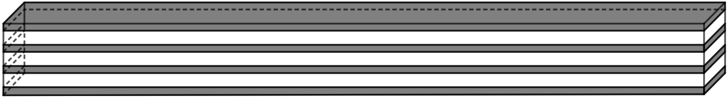
- The observation time $2C$ is not optimal. We are working towards using the generalized Ingham's theorem to bypass the low-frequency eigenvalues, which can not be described by the asymptotic analysis.
- Observability is a necessary condition for exp. stabilizability. Designing a controller to exp. stabilize the overall dynamics to the rest position is quite challenging by the multipliers method due to the non-compact perturbation operator J.
- Simulating the vibrations is computationally expensive. That is why we put this in the framework of a Wolfram Demonstration Project to consider all possible cases with different controllers.

Open Problems

- Order-reduction technique for hinged-hinged boundary conditions is an open problem. An interesting route is to adopt the non-harmonic Fourier series method and bypass the multipliers method.
- The Mead-Marcus beam model has been generalized for arbitrary number of layers. For $n = 2m + 1$ $(m > 1)$ layers,

$$\begin{cases} \ddot{z} + z'''' - \mathbf{B}^T \vec{\phi}' = 0, \\ -\mathbf{C}\vec{\phi}'' + \mathbf{P}\vec{\phi} = -\mathbf{B}z''', \quad (x,t) \in (0,L) \times \mathbb{R}^+ \end{cases}$$

where $\mathbf{B}$ is a $m \times 1$ column vector with positive entries, $\mathbf{P}$ and $\mathbf{C}$ are $m \times m$ invertible, symmetric, positive definite matrices.



Bibliography I

A.Ö. Özer, **A.K. Aydin**, Robust-Filtering of Sensor Data for the Finite Difference Model Reduction of a Piezoelectric Sandwich Beam, IEEE L-CSS, under review, 2022.

A.K. Aydin, M.Poynter, L. Stewart, A.Ö. Özer, Controlling vibrations on a three-layer beam by a single boundary controller, Wolfram Demonstration Project, preprint.

A.K. Aydin, T. Crouch, A.Ö. Özer, Implementation of the Fourier direct filtering technique to wave and beam equations, Wolfram Demonstration Project, preprint.

B-Z. Guo, J. Liu, A New Semidiscretized Order Reduction Finite Difference Scheme for Uniform Approximation of One-Dimensional Wave Equation, SIAM J. Control Optim., vol. 58, pp. 2256–2287, 2020.

Bibliography II

B-Z. Guo, J. Liu,, Uniformly Semidiscretized Approximation for Exact Observability and Controllability of One-dimensional Euler-Bernoulli Beam, Systems & Control Letters, 2021.

V. Komornik, *Exact Controllability and Stabilization. The Multiplier Method,* RAM. Masson-John Wiley, Paris; 1994.

V. Komornik, P. Loreti, *Fourier Series in Control Theory,* Springer, New York; 2005.

A.Ö. Özer, S.W. Hansen, Exact boundary controllability of an abstract Mead-Marcus sandwich beam model, 49th IEEE Conference on Decision and Control (CDC), 2010, pp. 2578–2583.

A.Ö. Özer, Exponential stabilization of a smart piezoelectric composite beam with only one boundary controller, Proc. of the 6th IFAC Symposium on Lagrangian and Hamiltonian Methods for Nonlinear Control, 2018, pp. 80–85.

Bibliography III

A.Ö. Özer, Uniform boundary observability of semi-discrete finite difference approximations of a Rayleigh beam equation with only one boundary observation, 58th IEEE Conference on Decision and Control (CDC), 2019, pp. 7708–7713.