# A New Semi-Discretization of the Fully Clamped Euler-Bernoulli Beam Preserving Boundary Observability Uniformly

**Joint Mathematics Meeting**

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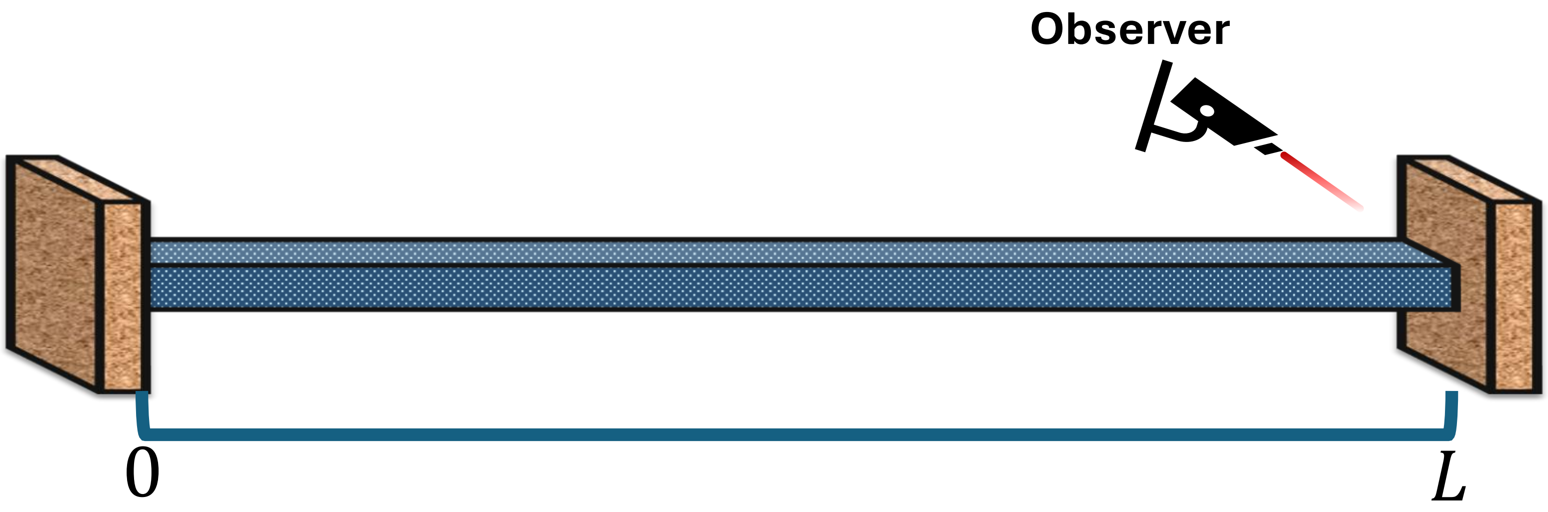
UMBC

# Outline

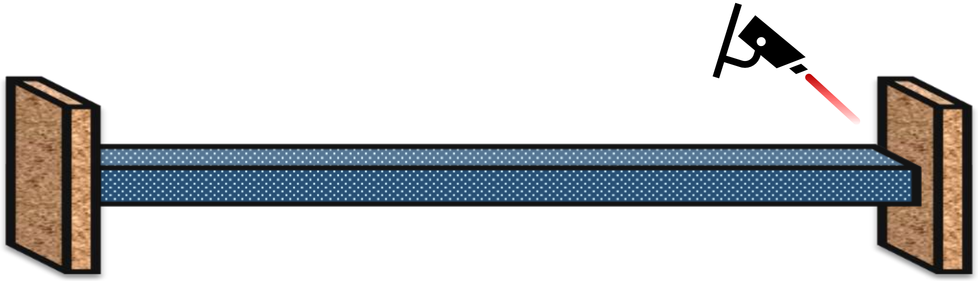
- **Continuous Problem**
  - Exact Observability
- **Order-Reduced Finite Differences**
  - Discretization
  - Energy of the system
- **Spectral Estimates**
  - Eigenvalue Problem
  - Spectral Gap
- **Uniform Observability**
- **Numerical Simulations**

# Euler Bernoulli Beam with Clamped BCs




$$
\begin{aligned}
&\ddot{z} + z'''' = 0, && (x,t) \in (0,L) \times \mathbb{R}^+ \\
&z(x,t) = z'(x,t)|_{x=0,L} = 0, && t \in \mathbb{R}^+ \\
&(z, \dot{z})(x,0) = (z_0, z_1)(x), && x \in [0,L]
\end{aligned}
$$

# Euler Bernoulli Beam with Clamped BCs



$$E(t) = \frac{1}{2}\int_0^L \left(|\dot{z}|^2 + |z''|^2\right)\,dx.$$

**Theorem:** For every $(z_0, z_1) \in H_0^2(0,L) \times L^2(0,L)$ and $T > 0$

- The model is well posed
- Energy is conserved i.e. $E(t) = E(0)$ for all $t \geq 0$.
- Exact observability inequality holds i.e. $\exists C(T) > 0$

$$\int_0^T |z''(L,t)|^2\,dt \geq C(T)E(0).$$

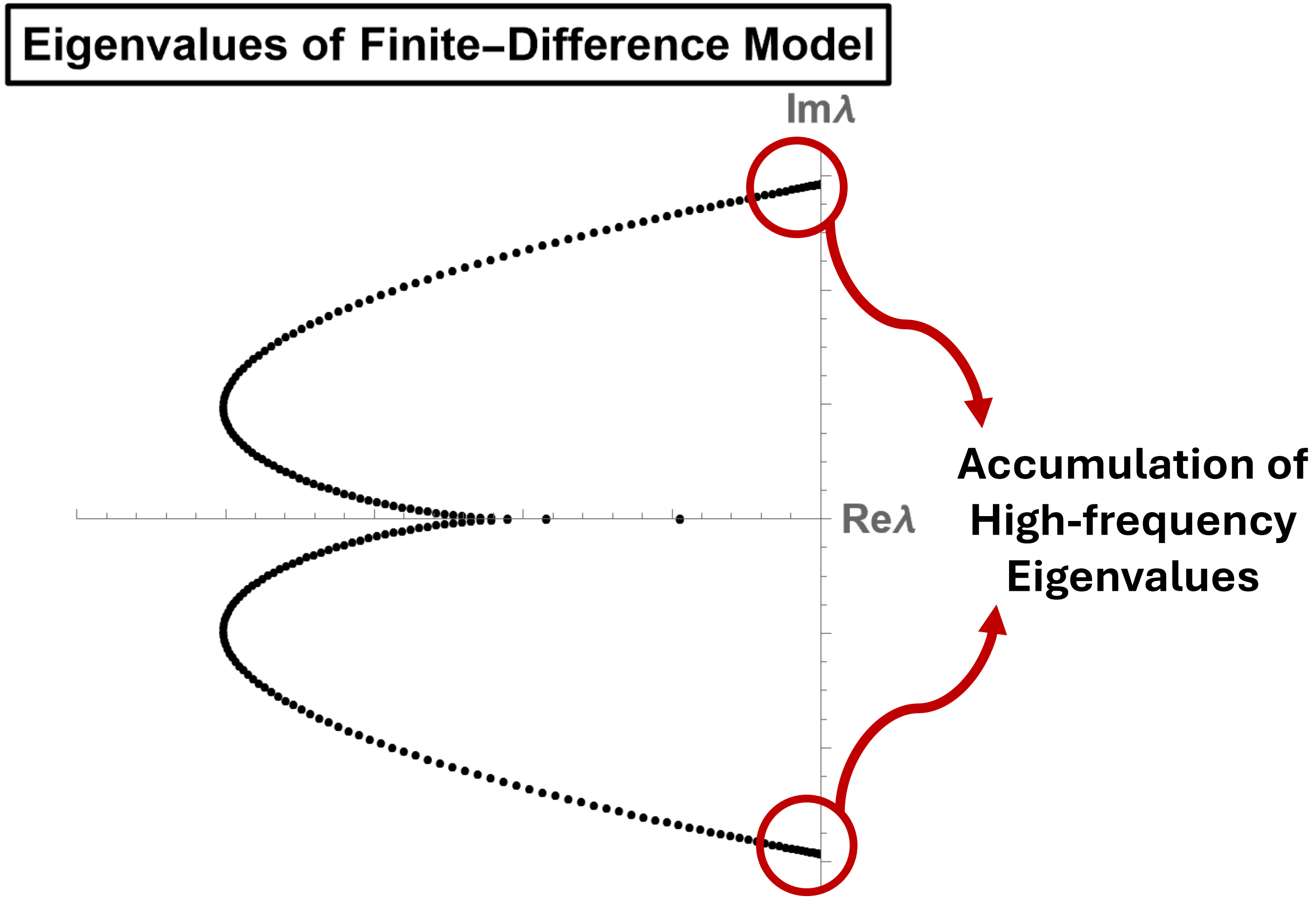
V. Komornik, Exact controllability and stabilization: the multiplier method, vol. 36. Elsevier Masson, 1994

# FD Lacks Exact Observability Uniformly

Finite Differences (FD) fails to retain Observability uniformly as $h \to 0$

**Control Problem:** High-frequency eigenvalues approach to imaginary axis

**Observability Problem:** High-frequency eigenvalues loses spectral gap condition




Feedback Controller: $z'(L,t) = -k\dot{z}(L,t)$

H.T. Banks, K. Ito, C. Wang, Exponentially stable approximations of weakly damped wave equations, in: Estimation and Control of Distributed Parameter Systems, Springer-Basel, 1991, 1–33

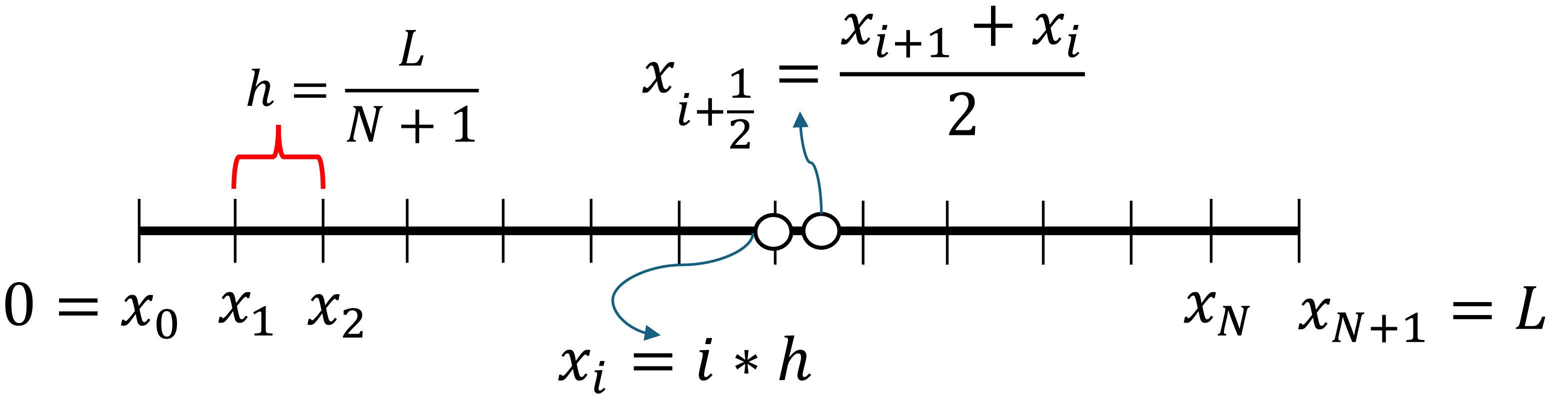
## Remedies to Retain Uniform Observability

- **Direct Fourier filtering:** Manually removes high-frequency modes
  Wave: [Infante, Zuazua ESAIM’99], EB: Clamped [Cindea, Micu, Roventa IFAC’16] , Hinged-Multilayer [Ozer, Aydin CDC’22]

- **Indirect Filtering:** Include a vanishing viscosity term
  Wave: [Tebou, Zuazua Adv. Comput. Math’07] EB-Wave(coupled): [Tebou MCRF’12]

- **Alternative Discretization Methods:**
  - Mixed Finite Elements (MFEM)
    Wave: [Castro, Micu Numer. Math’06] EB: [Cindea, Micu, Roventa SICON’16]
  - Order Reduced Finite Differences (ORFD)
    Wave: clamped-free-[Liu, Guo SICON’20],
    EB: cantilever-[Ozer, Aydin ESAIM’22], Hinged-Multilayer [Aydin, Ozer LCSS’23]

# ORFD Semi-Discretization




Consider the control intervals $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (\ddot{z} + z'''') \, dx = \underbrace{\frac{h}{2} \left( \ddot{z}(x_{i+\frac{1}{2}}, t) + \ddot{z}(x_{i-\frac{1}{2}}, t) \right)}_{\frac{h}{4} \left( \ddot{z}_{i-1} + 2\ddot{z}_i + \ddot{z}_{i+1} \right)} + z'''(x_{i+\frac{1}{2}}, t) - z'''(x_{i-\frac{1}{2}}, t) + \mathcal{O}(h^3)$$

## Difference Matrix

$$A = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix}$$

## Mass Matrix

$$M = \frac{1}{4}\begin{bmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \ddots & \vdots \\ 0 & 1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 2 \end{bmatrix}$$

## Boundary Nodes

$$D = \frac{2}{h^4}\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

# Matrix Formulation

$$\boldsymbol{\mathcal{A}_h} := \boldsymbol{A}^2 + \boldsymbol{D}$$

$$\begin{cases} \boldsymbol{M}\ddot{\vec{z}} + \boldsymbol{\mathcal{A}_h}\vec{z} = 0, & t \in \mathbb{R}^+ \\ (z, \dot{z})(0) = (z^0, z^1)(x_i), & i = 0, 1, \ldots, N+1 \end{cases}$$

The energy of the system is

$$\begin{aligned} E_h(t) &= \frac{h}{2}\left(||\boldsymbol{M}^{\frac{1}{2}}\dot{\vec{z}}||^2 + ||\boldsymbol{A}\vec{z}||^2 + ||\boldsymbol{D}^{\frac{1}{2}}\vec{z}||^2\right) \\ &= \frac{h}{2}\left(\langle \boldsymbol{M}\dot{\vec{z}}, \dot{\vec{z}}\rangle + \langle \boldsymbol{A^2}\vec{z}, \vec{z}\rangle + \langle \boldsymbol{D}\vec{z}, \vec{z}\rangle\right) \end{aligned}$$

# Conservation of Energy

The energy of the system is

$$E_h(t) = \frac{h}{2}\left(||\boldsymbol{M}^{\frac{1}{2}}\dot{\vec{z}}||^2 + ||\boldsymbol{A}\vec{z}||^2 + ||\boldsymbol{D}^{\frac{1}{2}}\vec{z}||^2\right)$$
$$= \frac{h}{2}\left(\langle \boldsymbol{M}\dot{\vec{z}}, \dot{\vec{z}}\rangle + \langle \boldsymbol{A^2}\vec{z}, \vec{z}\rangle + \langle \boldsymbol{D}\vec{z}, \vec{z}\rangle\right)$$

**Lemma:** The energy is conservative, i.e. $E_h(t) = E_h(0)$ for any $t > 0$.

**Proof:** $$\dot{E}_h(t) = \frac{h}{2}\left(\langle \boldsymbol{M}^{\frac{1}{2}}\ddot{\vec{z}}, \boldsymbol{M}^{\frac{1}{2}}\dot{\vec{z}}\rangle + \langle \boldsymbol{A}\dot{\vec{z}}, \boldsymbol{A}\vec{z}\rangle + \langle \boldsymbol{D}^{\frac{1}{2}}\dot{\vec{z}}, \boldsymbol{D}^{\frac{1}{2}}\vec{z}\rangle\right)$$
$$= \frac{h}{2}\langle \boldsymbol{M}\ddot{\vec{z}} + (\boldsymbol{A}^2 + \boldsymbol{D})\vec{z}, \dot{\vec{z}}\rangle = \frac{h}{2}\langle \boldsymbol{M}\ddot{\vec{z}} + \boldsymbol{\mathcal{A}_h}\vec{z}, \dot{\vec{z}}\rangle.$$

# Conservation of Energy

The energy of the system is

$$E_h(t) = \frac{h}{2}\left(||\boldsymbol{M}^{\frac{1}{2}}\dot{\vec{z}}||^2 + ||\boldsymbol{A}\vec{z}||^2 + ||\boldsymbol{D}^{\frac{1}{2}}\vec{z}||^2\right)$$
$$= \frac{h}{2}\left(\langle \boldsymbol{M}\dot{\vec{z}}, \dot{\vec{z}}\rangle + \langle \boldsymbol{A}^{\boldsymbol{2}}\vec{z}, \vec{z}\rangle + \langle \boldsymbol{D}\vec{z}, \vec{z}\rangle\right)$$

**Lemma:** The energy is conservative, i.e. $E_h(t) = E_h(0)$ for any $t > 0$.

**Lemma:** Equipartition of Energy

$$\langle \boldsymbol{M}^{\frac{1}{2}}\dot{\vec{z}}, \boldsymbol{M}^{\frac{1}{2}}\dot{\vec{z}}\rangle = \langle \boldsymbol{A}\vec{z}, \boldsymbol{A}\vec{z}\rangle + \langle \boldsymbol{D}^{\frac{1}{2}}\vec{z}, \boldsymbol{D}^{\frac{1}{2}}\vec{z}\rangle$$

# Solution Steps

Eigenvalue problem:

$$\boldsymbol{A_h}\vec{\psi} = \mu \boldsymbol{M}\vec{\psi}$$

Fourier Expansion of the solution:

$$\vec{z}(t) = \sum_{|k|=1}^{N} \alpha_k e^{-i\lambda_k t} \frac{\psi_{|k|}}{\lambda_k}$$

$$\lambda_k = \operatorname{sgn}(k)\sqrt{\mu_{|k|}}$$

**Haraux's Theorem** Let $\{\lambda_k\}$ satisfy

uniform gap $\gamma := \inf_{m \neq n} |\lambda_m - \lambda_n| > 0$

and $\gamma' := \sup_{A \subset K} \inf_{\substack{m,n \in K \setminus A \\ m \neq n}} |\lambda_m - \lambda_n|$

For $T > \frac{2\pi}{\gamma'}$, there exist $c_1, c_2 > 0$ such that

$$\boxed{c_1 \sum_{k \in K} |a_k|^2 \leq \int_I |z(t)|^2 dt \leq c_2 \sum_{k \in K} |a_k|^2}$$

for all functions

$$z(t) = \sum_{k \in K} a_k e^{i\lambda_k t}$$

# Spectral Gap

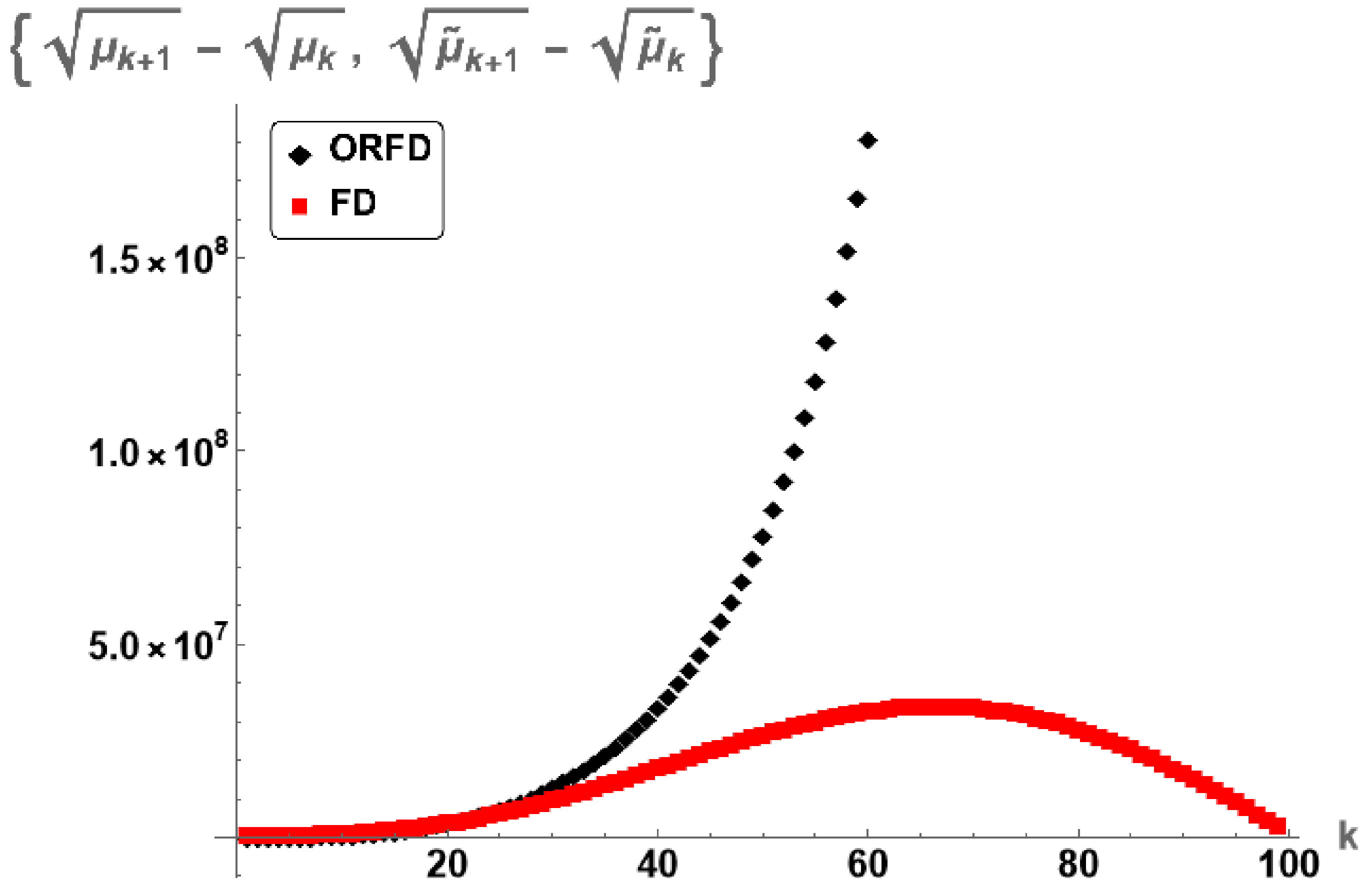



**Figure:** For $N = 100$, the gap among the consecutive eigenvalues, corresponding to the Order-Reduced Finite Difference (ORFD) and the Finite Difference (FD) model reductions, respectively.

# Eigenvalues of Matrices

$$\lambda_k(\boldsymbol{A}) = \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2L}\right)$$

$$\boldsymbol{M} = \boldsymbol{I} - \frac{h^2}{4}\boldsymbol{A}$$

$$\lambda_k(\boldsymbol{M}) = 1 - \sin^2\left(\frac{k\pi h}{2L}\right) = \cos^2\left(\frac{k\pi h}{2L}\right)$$

$$\lambda_k(\boldsymbol{M}^{-1}\boldsymbol{A}^2) = \frac{16}{h^4}\sin^2\left(\frac{k\pi h}{2L}\right)\tan^2\left(\frac{k\pi h}{2L}\right)$$

# Spectral Gap

**Lemma:** For any $\gamma' > 0$ there exist $n_\gamma \in \mathbb{N}$ such that

$$\sqrt{\mu_{k+1}} - \sqrt{\mu_k} \geq \gamma', \quad \forall k \geq n_\gamma$$

**Proof:**

$$\lambda_k(\boldsymbol{M}^{-1}\boldsymbol{A}^2) = \frac{16}{h^4}\sin^2\left(\frac{k\pi h}{2L}\right)\tan^2\left(\frac{k\pi h}{2L}\right)$$

by the Weyl's inequality

$$\lambda_k(M^{-1}\boldsymbol{A}^2) \leq \lambda_k(M^{-1}\boldsymbol{\mathcal{A}}_h) \leq \lambda_k(M^{-1}\boldsymbol{A}^2) + \lambda_{\max}(M^{-1}\boldsymbol{D})$$

where $\lambda_{\max}(\boldsymbol{M}^{-1}\boldsymbol{D}) \leq \frac{4}{h^4}$

$$\sqrt{\mu_{k+1}} - \sqrt{\mu_k} \geq \sqrt{\lambda_{k+1}(M^{-1}\boldsymbol{A}^2)} - \sqrt{\lambda_k(M^{-1}\boldsymbol{A}^2)} - \frac{2}{h^2} \to \infty$$

# Main Result

**Theorem (Uniform Observability):** Let $T > 0$. There exists $N_0(T) \in \mathbb{N}$ such that, for any $N > N_0$ there exist $C_1, C_2 > 0$ independent of $h$ such that following observability inequality holds

$$C_1 E_h(t) \leq L \int_0^T \left( \left| \frac{z_N}{h^2} \right|^2 + |\dot{z}_N|^2 \right) dt \leq C_2 E_h(t).$$

**Remark:** The two observers is consistent with the uniform observability for the wave equation [Castro, Micu'06] The observers shown to converge to the controller of continuous model.

# Sketch of the Proof

**Lemma:** Let $\vec{\psi}$ be an eigenvector. Then

$$\frac{67h}{16}\langle \boldsymbol{M}\vec{\psi}, \vec{\psi}\rangle \geq \left(\frac{L}{4} + \frac{L}{h^4\mu}\right)\psi_N^2 \geq \frac{h}{2}\langle \boldsymbol{M}\vec{\psi}, \vec{\psi}\rangle$$

**Proof:** Technical proof done by using multiplier $i\,(\psi_{i+1} - \psi_{i-1})$.

# Sketch of the Proof

$$\boxed{\frac{67h}{16}\langle \boldsymbol{M}\vec{\psi},\vec{\psi}\rangle \geq \left(\frac{L}{4}+\frac{L}{h^4\mu}\right)\psi_N^2 \geq \frac{h}{2}\langle \boldsymbol{M}\vec{\psi},\vec{\psi}\rangle}$$

**Equipartition of the Energy**

$$E(t) = E(0) = \sum_{|k|=1}^{N} |\alpha_k|^2 h \langle \boldsymbol{M}\vec{\psi}_{|k|},\vec{\psi}_{|k|}\rangle$$

$$\int_0^T |\dot{z}_{N+\frac{1}{2}}|^2 dt = \frac{1}{4}\int_0^T \sum_{|k|=1}^{N} |\alpha_k e^{-i\lambda_k t}|^2 |\psi_{|k|,N}|^2$$

$$\int_0^T \left|\frac{z_N}{h^2}\right|^2 dt = \underbrace{\int_0^T \sum_{|k|=1}^{N} |\alpha_k e^{-i\lambda_k t}|^2}_{\text{Haraux's Theorem}} \left|\frac{\psi_{|k|,N}}{h^2\sqrt{\mu_{|k|}}}\right|^2$$

# Numerical Simulations

**Observability**

$$z_{N+2} = 0$$

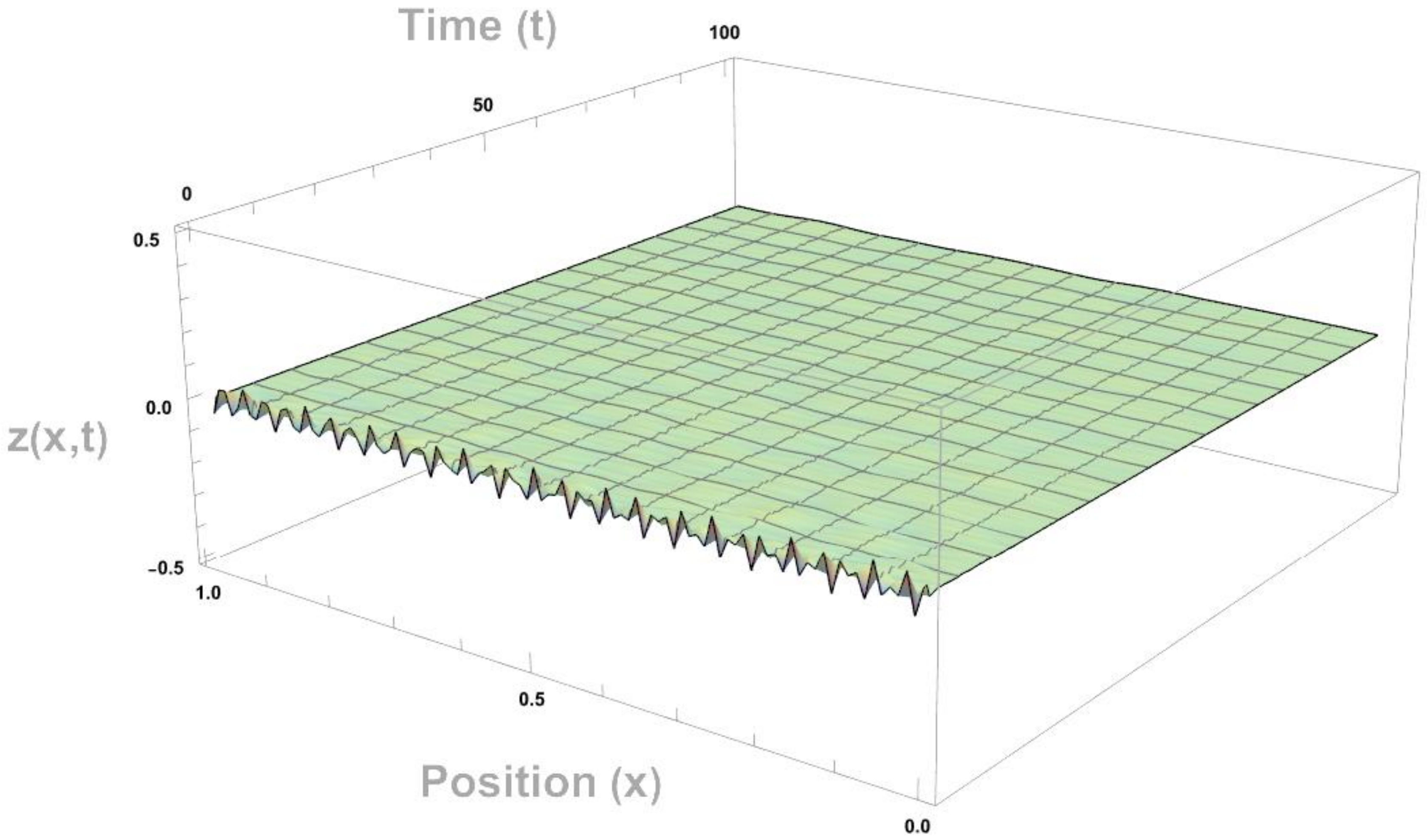
**Control**

$$z_{N+2} = -k_1 h \dot{z}_N - k_2 \frac{z_N}{h}$$

$$\boldsymbol{B} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

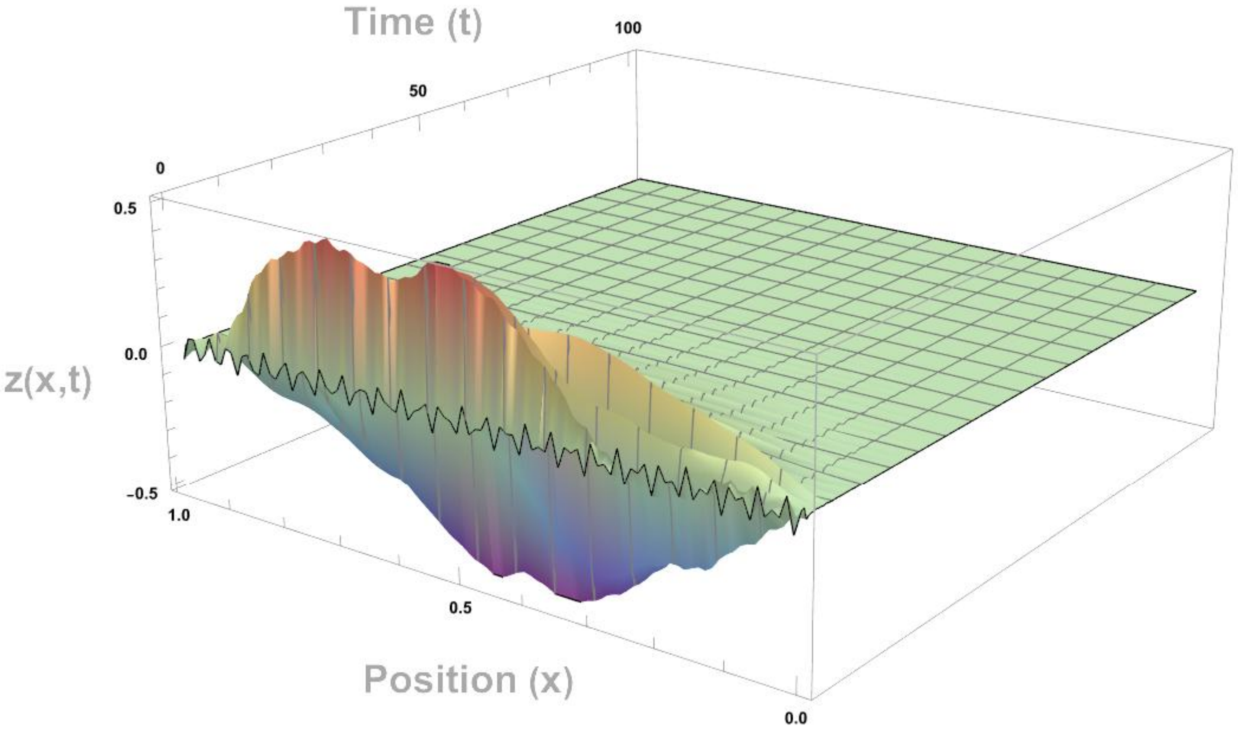
$$\boldsymbol{M}\ddot{\vec{z}} + \boldsymbol{A_h}\vec{z} + \frac{k_1}{h^3}\boldsymbol{B}\dot{\vec{z}} + \frac{k_2}{h^5}\boldsymbol{B}\vec{z} = 0,$$

# Numerical Simulations

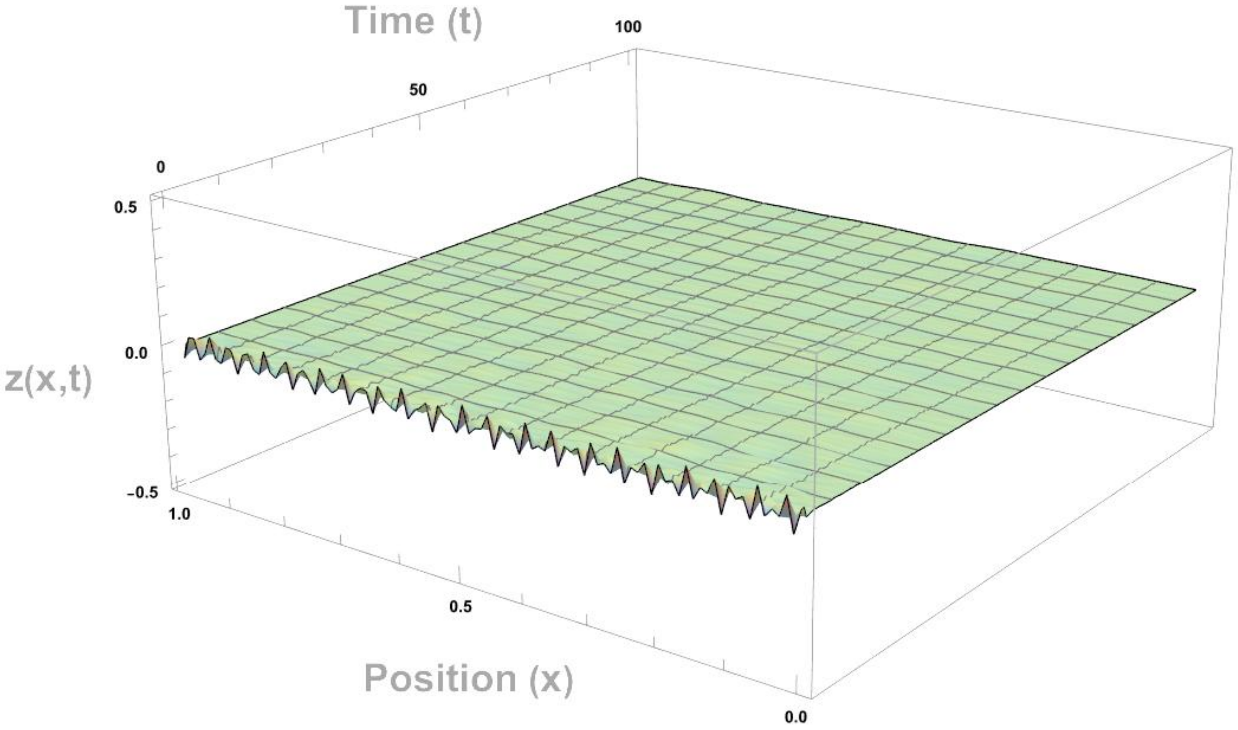


# Eigenvalues

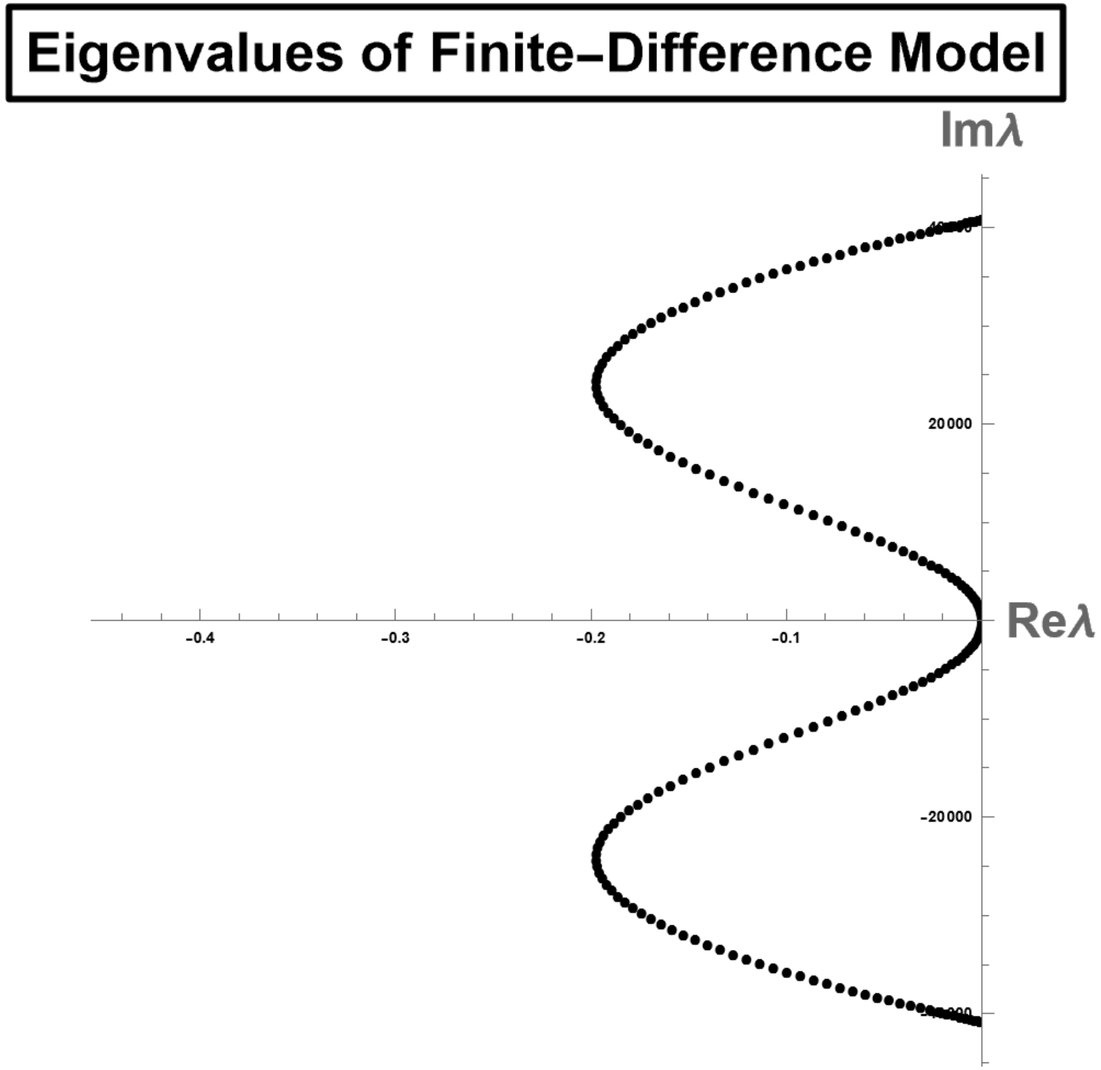
$$k_1 \neq 0 \text{ and } k_2 = 0$$

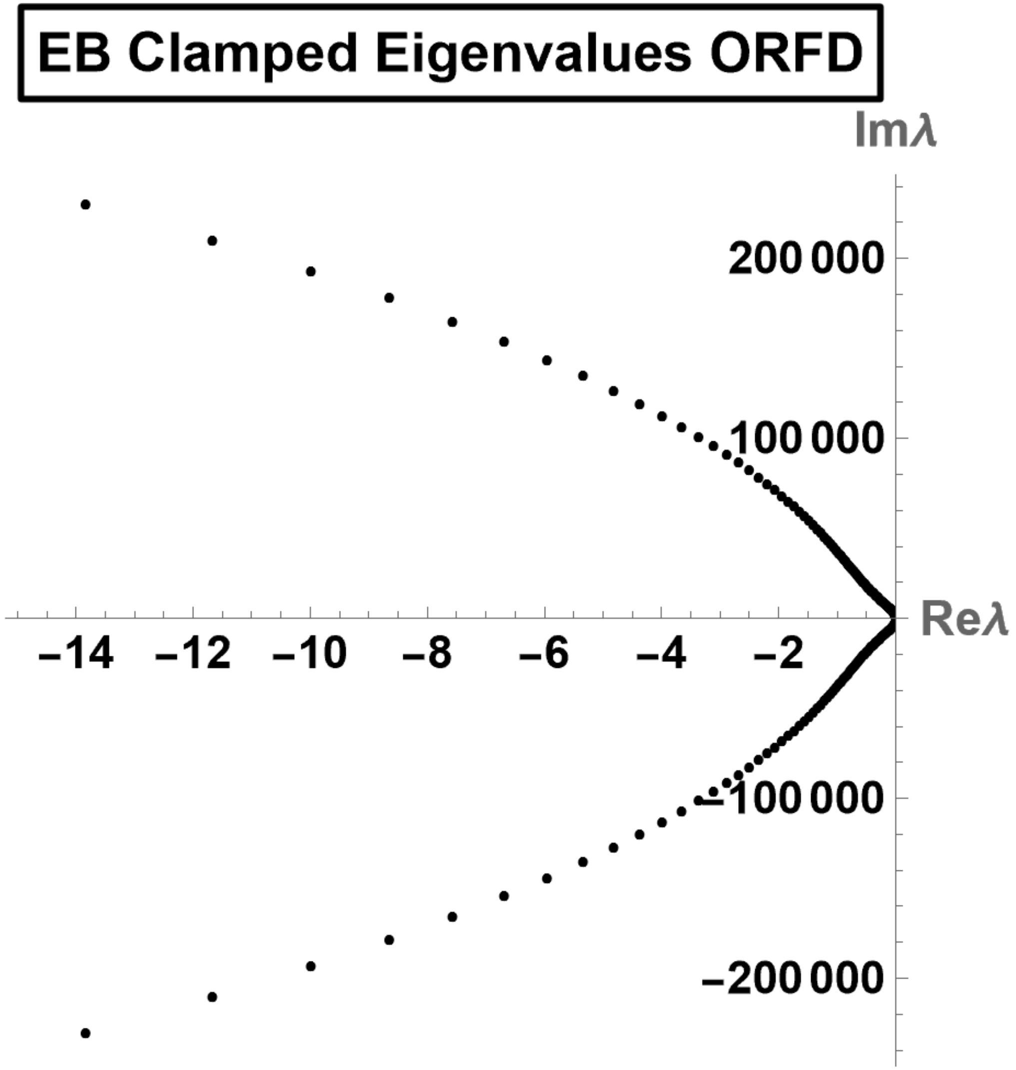



$$k_1 \neq 0 \text{ and } k_2 \neq 0$$



# ORFD vs FD Eigenvalues of Control Problem



# Ongoing & Future Work

- **Control Problem:**
  - Proving uniform controllability & exponential stability
  - Convergence of the dual controllers to the HUM controller

- **Extension of the Model:**
  - Multilayered Beam models
  - Serially connected models
    - A New SCOLE-type Hybrid Beam Network With Joint Dynamics And A Novel Mixed-order Boundary-interface Control Strategy, Akil, Fragnelli, Ismail, Ozer, Submitted.
    - On the Unconditional Stability of Serially Connected SCOLE-Type Beam Systems with Higher- and Lower-Order Joint Damping, , Akil, Ashburn, Brown, Fragnelli, Ismail, Ozer. submitted

- **I have another presentation:**
  - **Tuesday 4:00 pm Room 304**

**THANK YOU!**