# A Novel Finite Difference-Based Model Reduction and a Sensor Design for a Multilayer Smart Beam with Arbitrary Number of Layers

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# Outline I

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   - Modeling Assumptions
   - PDE Model
     - Sensor Design (Uniform Observability) in PDE setting
2. Standard Finite Difference Model Reduction
   - Extremely Close Vibrational Frequencies- Sensor Failure
   - Recovering Observability with Filtering - Robust Sensor Design
3. Order Reduced Finite Difference Model Reduction
   - Uniform Observability as $h \to 0$
4. Simulations & Wolfram Demonstration Projects
5. Conclusions, Ongoing Work, and Future Directions

# Generalized Multi-layer Structure

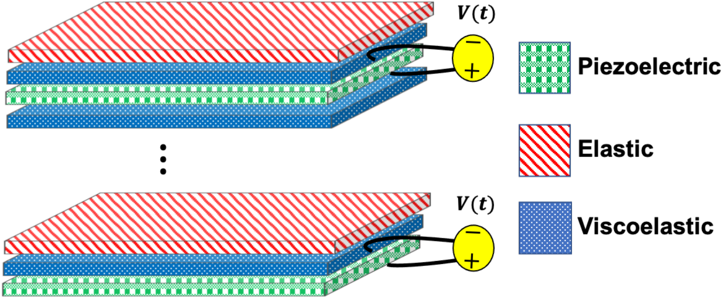


Figure: Multi-layer beam model with alternating $m+1$ elastic/piezoelectric and $m$ viscoelastic Layers. Piezoelectric layers are attached to electric circuits.

# Sandwich Modeling Assumptions

- No independent longitudinal vibration dynamics across layers
- Only uniform transversal (bending) vibrations accross layers
- The layers are **perfectly bonded**, such that no slip occurs.

- The outer elastic/piezoelectric layers, resisting transverse shear, assume the Euler-Bernoulli small displacement assumptions.
- The sandwiched viscoelastic layers, allowing transverse shear, hold the Mindlin-Timoshenko small displacement assumptions.

- The resulting PDE models (**Mead-Marcus** or Rao-Nakra) are **LINEAR**.

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R.H. Fabiano and S.W. Hansen, *Modeling and analysis of a three-layer damped sandwich beam*, Discrete Contin. Dynam. Syst., Added Volume (2001), 143-155.

A. Ö Özer, *Exponential stabilization of a smart piezoelectric composite beam with only one boundary controller*, Proc. of the 6th IFAC Symposium, Valparaíso, Chile, IFAC, 2018, 51-3, 80–85.

A. Ö Özer, *Dynamic and non-dynamic modeling for a piezoelectric smart beam and related preliminary stabilization results*, Evolution Equations and Control Theory, (7-4), 2018, 639-668.

# Sensor Failure $\Longrightarrow$ Actuator Design Failure

- $\boxed{\text{Blind model reductions cause uncontrollability.}}$

# A Mead-Marcus-type Multi-layer beam model

- $z(x,t)$ : Uniform bending of the centerline of the laminate.
- $\vec{\phi} = \begin{bmatrix}\phi_1 & \cdots & \phi_m\end{bmatrix}^T$ is an $m \times 1$ **column vector** where $\phi_i$ is the shear angle of $i^{th}$ layer.

$$\begin{cases} \ddot{z} + z'''' - \boldsymbol{B}^T \vec{\phi}' = 0, & \\ -\boldsymbol{C}\vec{\phi}'' + \boldsymbol{P}\vec{\phi} = -\boldsymbol{B}z''', & (x,t) \in [0,L] \times \mathbb{R}^+ \\ z, z''\ \vec{\phi}'\Big|_{x=0,L} = 0, & t \in \mathbb{R}^+ \\ (z,\dot{z})(x,0) = (z_0, z_1)(x), & x \in [0,L]. \end{cases}$$

- $\boldsymbol{B}$ is an $m \times 1$ **column vector** with positive entries,
- $\boldsymbol{P}$ and $\boldsymbol{C}$ are $m \times m$ invertible, symmetric, positive-definite **matrices**.

# Simplifying the PDE Model

- $\mathrm{Dom}\left(D_x^2 = \frac{\partial^2}{\partial x^2}\right) := \{\phi \in H^2(0,L) \mid \phi_x(0) = \phi_x(L) = 0\}.$
- Since the operator $(-\boldsymbol{C}D_x^2 + \boldsymbol{P}) : \mathrm{Dom}(D_x^2) \to L^2(0,L)$ is boundedly invertible, the operator $\boldsymbol{J} := (-\boldsymbol{C}D_x^2 + \boldsymbol{P})^{-1}$ exists and bounded.

$$\begin{cases} \ddot{z} + z'''' - \boldsymbol{B}^T\vec{\phi}' = 0, & \\ \boxed{-\boldsymbol{C}\vec{\phi}'' + \boldsymbol{P}\vec{\phi} = -\boldsymbol{B}z''',} & (x,t) \in (0,L) \times \mathbb{R}^+ \\ z, z'', \vec{\phi}'\Big|_{x=0,L} = 0, & t \in R^+ \\ (z,\dot{z})(x,0) = (z^0, z^1)(x). & x \in [0,L] \end{cases}$$

- The second equation is **elliptic** and can be solved in terms of $z(x,t)$.

$$\boxed{\vec{\phi}(x,t) = \left[-(-\boldsymbol{C}D_x^2 + \boldsymbol{P})^{-1}\boldsymbol{B}z'''\right](x,t) = -\boldsymbol{J}\boldsymbol{B}z'''(x,t).}$$

# Total Energy - What is Observed on the Beam?

The natural energy is

$$E_{\mathrm{nat}}(t) = \frac{1}{2}\int_0^L \left[ \underbrace{|\dot{z}|^2}_{\text{Kinetic}} + \underbrace{|z''|^2}_{\text{Bending Potential}} + \underbrace{(\boldsymbol{B}^T \boldsymbol{J} \boldsymbol{B} z''')(z')}_{\text{Shear Potential}} \right] dx.$$

The higher-order energy $E(t)$ is defined by

$$E(t) = \|(z,\dot{z})\|_E^2 = \frac{1}{2}\int_0^L \left[ |\dot{z}'|^2 + |z'''|^2 + (\boldsymbol{B}^T \boldsymbol{J} \boldsymbol{B} z''')(z''') \right] dx,$$

where $\boldsymbol{J} := (-CD_x^2 + P)^{-1}$.The (M-M) beam model is conservative, *i.e.* $\frac{dE(t)}{dt} = 0$, therefore

$$\boxed{E(0) = E(t)}, \quad \forall t \geq 0.$$

# What has been done for the Exact Observability?

- Let $\mathcal{H} := H^3(0,L) \cap H_0^1 \times H_0^1(0,L)$.

**GOAL:** Prove that for each $(z^0, z^1) \in \mathcal{H}$, there exists a constant $C(T) > 0$ such that for $T > T_{MIN} = 0$, the so-called exact observability inequality holds

$$\int_0^T |\dot{z}'(L,t)|^2 dt \geq C(T)E(0) = C(T)\|(z^0, z^1)\|_E^2.$$

## Remark

*Exact controllability (dual problem) for a* **three-layer beam** *is proved in (Rajaram & Hansen'03) by the Method of Moments. This automatically leads to an observability inequality yet it*

- *does not state the sensor measurement explicitly*
- *is not generalized to arbitrary number of layers*
- *does not consider model reductions*

- [Aydin,Özer-IEEE-CDC'22]: Proved for **arbitrary number of layers.**

# Uniform Observability (Sensor Design)

**Theorem (Aydin-Özer, L-CSS'23, IEEE-CDC'23)**

*For any* $T > 0$ *there exist* $C_1(T), C_2(T) > 0$ *such that*

$$C_1(T)E(0) \geq \int_0^T |\dot{z}'(L,t)|^2 dt \geq C_2(T)E(0).$$

- The proof uses a Spectral Analysis (Non-harmonic Fourier series)

- One sensor measuring only $\dot{z}'(L,t)$: The angular velocity at the TIP $x = L$.

# Standard Finite Difference Model Reduction

Let $N \in \mathbb{N}$ be given, and define the mesh parameter $h := \frac{L}{N+1}$. Consider the following discretization of the interval $[0, L]$:

$$0 = x_0 < x_1 < \dots < x_j = jh < \dots < x_{N-1} < x_N < x_{N+1} = L.$$

Let $z_j = z_j(t) \approx z(x_j, t)$, and $\vec{z} = [z_1, z_2, \dots, z_N]^T$.

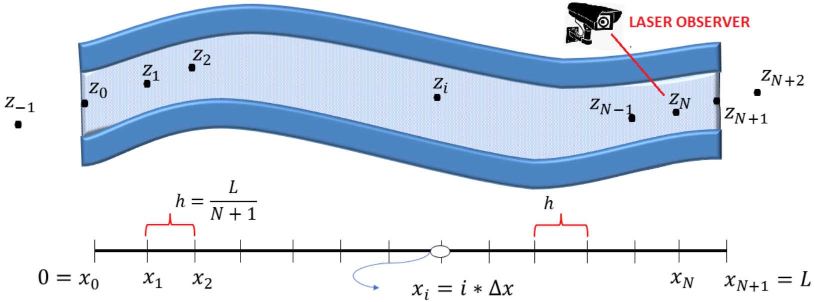



Figure: The laser sensor observes the angle of velocity at the tip of the beam.

# A Special Positive-Definite Tridiagonal Matrix $A_h$

Consider the Finite-Difference approximation of $\frac{\partial^2}{\partial x^2}$ by central differences:

$$z''(x \to x_j, t) \approx \frac{z_{j+1} - 2z_j + z_{j-1}}{h^2}.$$

The corresponding matrix is

$$\boldsymbol{A}_h = \frac{1}{h^2} \begin{bmatrix} 2 & -1 & 0 & \dots & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & -1 & 2 & -1 & 0 \\ 0 & \dots & \dots & 0 & -1 & 2 & -1 \\ 0 & \dots & \dots & \dots & 0 & -1 & 2 \end{bmatrix}$$

whose eigen-pairs $(\lambda_k(h), \vec{\varphi}_k(h))$ are

$$\begin{cases} \lambda_k(h) = \frac{4}{h^2} \sin^2\left(\frac{k\pi h}{2L}\right), \\ \varphi_{k,j} = \sin\left(\frac{jk\pi h}{L}\right), \quad k, j = 1, 2, ..., N. \end{cases} \tag{1}$$

# The Reduced Model and the Total Energy

For $\vec{z} = [z_1, \ldots x_N]$, the space semi-discretization of **(M-M)**

$$\begin{cases} \ddot{\vec{z}} + \boldsymbol{A}_h^2 \vec{z} + \boldsymbol{J}_h \boldsymbol{A}_h^2 \vec{z} = 0, \\ z_0 = z_{N+1} = 0, \\ z_{-1} = -z_1, \;\; z_{N+2} = -z_N, \quad t \in \mathbb{R}^+ \end{cases}$$

can be reformulated in the first-order form

$$\frac{\partial}{\partial t} \begin{bmatrix} \vec{z} \\ \dot{\vec{z}} \end{bmatrix} = \begin{bmatrix} 0_{N\times N} & \boldsymbol{I}_{N\times N} \\ -(\boldsymbol{I} + \boldsymbol{J}_h)\boldsymbol{A}_h^2 & 0_{N\times N} \end{bmatrix} \begin{bmatrix} \vec{z} \\ \dot{\vec{z}} \end{bmatrix} := \mathcal{A}_{\mathrm{FD}} \begin{bmatrix} \vec{z} \\ \dot{\vec{z}} \end{bmatrix}.$$

where $\boldsymbol{J}_h := (\boldsymbol{B}^T \otimes \boldsymbol{I})(\boldsymbol{C} \otimes \boldsymbol{A}_h + \boldsymbol{P} \otimes \boldsymbol{I})^{-1}(\boldsymbol{B} \otimes \boldsymbol{I})$, and $\otimes$ is the Kronecker product.

For $\vec{z} = [z_1, \ldots x_N]$, the space semi-discretization of **(M-M)**

$$\begin{cases} \ddot{\vec{z}} + \boldsymbol{A}_h^2 \vec{z} + \boldsymbol{J}_h \boldsymbol{A}_h^2 \vec{z} = 0, \\ z_0 = z_{N+1} = 0, \\ z_{-1} = -z_1, \;\; z_{N+2} = -z_N, \quad t \in \mathbb{R}^+ \end{cases}$$

The discretized energy is

$$E_h(t) = \frac{h}{2} \sum_{j=1}^{N} \left| \frac{\dot{z}_{j+1} - \dot{z}_j}{h} \right|^2 + \left| \frac{(\boldsymbol{A}_h \vec{z})_{j+1} - (\boldsymbol{A}_h \vec{z})_j}{h} \right|^2 + \left( \frac{(\boldsymbol{J}_h \boldsymbol{A}_h \vec{z})_{j+1} - (\boldsymbol{J}_h \boldsymbol{A}_h \vec{z})_j}{h} \right) \left( \frac{(\boldsymbol{A}_h \vec{z})_{j+1} - (\boldsymbol{A}_h \vec{z})_j}{h} \right)$$

where $\boldsymbol{J}_h := (\boldsymbol{B}^T \otimes \boldsymbol{I})(\boldsymbol{C} \otimes \boldsymbol{A}_h + \boldsymbol{P} \otimes \boldsymbol{I})^{-1}(\boldsymbol{B} \otimes \boldsymbol{I})$, and $\otimes$ is the Kronecker product.

# Explicit Representation of Eigen-pairs

## Lemma (Aydin, Özer, Walterman, IEEE-LCSS'23 )

*Eigenvalues $\mu_k$ of $(\boldsymbol{I} + \boldsymbol{J}_h)\boldsymbol{A}_h^2$ defined as for $k = 1, 2, ..., N$,*

$$\mu_k(h) = (1 + \omega_k(h))\, \lambda_k^2(h),$$

$$\omega_k(h) = \frac{(\boldsymbol{B}^T\boldsymbol{B})^2}{\lambda_k(h)\boldsymbol{B}^T\boldsymbol{C}\boldsymbol{B} + \boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B}}$$

*with $\sqrt{\mu_k(h)} := -\sqrt{\mu_{-k}(h)}$ for $k = -1, -2, ..., -N$. Then, the solutions can be expressed as*

$$\vec{z}(t) = \sum_{k \in K} (a_k e^{i\sqrt{\mu_k(h)}t})\vec{\varphi}_k(h).$$

# Sensor Failure by this Model Reduction

## Theorem (Aydin-Özer, L-CSS'23, IEEE-CDC'23)

*[Infinite Observation Time-Redundancy] For any* $T > 0$,

$$\lim_{h \to 0} \sup_{\vec{z} \text{ solves } (\mathrm{M-M})} \frac{E_h(0)}{\int_0^T \left| \frac{\dot{z}_N}{h} \right|^2 dt} \to \infty$$

## Sketch of the Proof.

Consider the highest-frequency eigen-solution, i.e. corresponding to $\vec{z}(0) = \vec{\varphi}_N$. By the identity (Infante-Zuazua '99)

$$\underbrace{h \sum_{j=0}^{N} \left| \frac{\varphi_{N,j+1} - \varphi_{N,j}}{h} \right|^2}_{\text{Energy}} = \underbrace{\frac{2L}{4 - \lambda_N(h) h^2}}_{\to\infty(\ h \to 0)} \underbrace{\left| \frac{\varphi_{N,N}}{h} \right|^2}_{\text{Sensor Measurement}}, \qquad (2)$$

$$\boxed{\lambda_N(h) h^2 = 4 \sin^2\left(\frac{N\pi h}{2L}\right) = 4 \cos^2\left(\frac{\pi h}{2L}\right) \to 4 \text{ as } h \to 0.} \qquad \square$$

# Direct Fourier Filtering - Before & After

$$\boxed{\gamma \in (0,4) : \textit{Filtering Parameter such that } \lambda_k h^2 \leq \gamma}$$

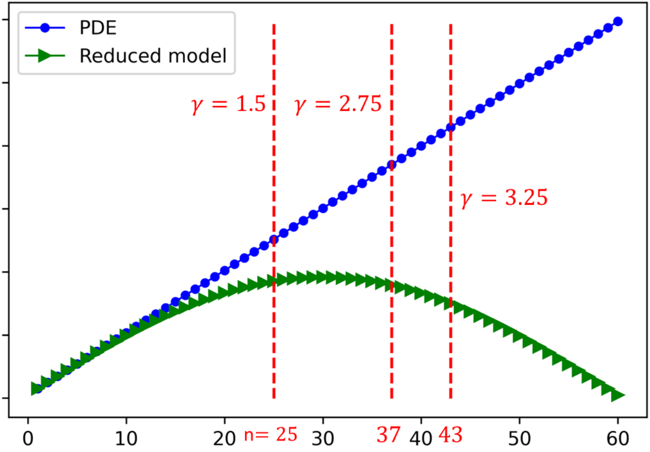



Figure: Spectral gap $|\mu_{n+1} - \mu_n|$, and $|\mu_{n+1}(h) - \mu_n(h)|$ for $n = 1, ..., 60$.

# Filtered Solutions Space & Recovering Observability

Given any $\gamma \in (0,4)$ , define the following FILTERED SOLUTIONS space

$$\boldsymbol{F}_h(\gamma) = \left\{ \vec{z}_h(t) \text{ sol. of } \textbf{(M-M)} \mid \vec{z}_h(t) = \sum_{\lambda_k(h) h^2 \leq \gamma} (a_k e^{i\mu_k(h)t}) \vec{\varphi}^k(h) \right\}.$$

**Theorem (Aydin, Özer, Walterman, IEEE-LCSS'23 )**

*Let $\gamma \in (0,4)$. For any $T > 0$, there exist $C = C(T,\gamma) > 0$ such that*

$$\int_0^T \left| \frac{\dot{z}_N}{h} \right|^2 dt \geq C(T,\gamma) E(0), \qquad \forall \vec{z}_h(t) \in \boldsymbol{F}_h(\gamma).$$

# Reduced Model

A novel discretization of the PDE model is proposed [2]:

$$\begin{cases} \frac{1}{4}\left(\ddot{z}_{j+1}+2\ddot{z}_j+\ddot{z}_{j-1}\right)+\delta_x^4 z_j+\frac{1}{4}\left(y_{j+1}+2y_j+y_{j-1}\right)=0, \\ z_0=0,\ z_{N+1}=0\ z_1=-z_{-1},\ z_{N+2}=-z_N, \\ z_j(0)=z^0(x_j), \quad \dot{z}_j(0)=z^1(x_j), \end{cases}$$

where $\vec{y} := \boldsymbol{J}_h \boldsymbol{A}_h^2 \vec{z}$. (18) can be reformulated as

$$\frac{\partial}{\partial t}\begin{bmatrix}\vec{z}\\ \dot{\vec{z}}\end{bmatrix}=\begin{bmatrix}0_{N\times N} & I_{N\times N}\\ -(\boldsymbol{M}^{-1}+\boldsymbol{J}_h)\boldsymbol{A}_h^2 & 0_{N\times N}\end{bmatrix}\begin{bmatrix}\vec{z}\\ \dot{\vec{z}}\end{bmatrix}:=\mathcal{A}_{\mathrm{OR}}\begin{bmatrix}\vec{z}\\ \dot{\vec{z}}\end{bmatrix}.$$

[2] A.K. Aydin, A.Ö. Özer, J. Walterman, *A Novel Finite Difference-based Model Reduction and a Sensor Design for a Multilayer Smart Beam with Arbitrary Number of Layers,* IEEE Control Systems Letters, vol. 7 (2023), 1548-1553.

# Special Mass Matrix

$$\boldsymbol{M} := \frac{1}{4}\begin{bmatrix} 2 & 1 & 0 & \dots & \dots & \dots & 0 \\ 1 & 2 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 2 & 1 & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 & 2 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 & 2 & 1 \\ 0 & \dots & \dots & \dots & 0 & 1 & 2 \end{bmatrix} = \frac{1}{4}(4\boldsymbol{I} - h^2\boldsymbol{A}_h)$$

Notice that the eigenvalues of $\boldsymbol{M}$ are

$$\tilde{\lambda}_k(h) = \frac{1}{4}\boxed{(4 - h^2\lambda_k(h))}.$$

# Discretization of Energy

The discretized energy

$$E_h(t) := \frac{h}{2}\sum_{j=1}^{N}\left(\frac{(\boldsymbol{M}\boldsymbol{J}_h\boldsymbol{A}_h^2\vec{z})_{j+1}-(\boldsymbol{M}\boldsymbol{J}_h\boldsymbol{A}_h^2\vec{z})_j}{h}\right)\left(\frac{\vec{z}_{j+1}-\vec{z}_j}{h}\right) + \left|\frac{(\boldsymbol{M}^{1/2}\dot{\vec{z}})_{j+1}-(\boldsymbol{M}^{1/2}\dot{\vec{z}})_j}{h}\right|^2 + \left|\frac{(\boldsymbol{A}_h z)_{j+1}-(\boldsymbol{A}_h z)_j}{h}\right|^2$$

is conservative, i.e., $\frac{dE_h(t)}{dt} = 0$. Hence, $E_h(t) = E_h(0)$ for all $t > 0$. Moreover, $\boldsymbol{M}^{1/2}$ is uniquely defined since $\boldsymbol{M}$ is positive definite.

# Uniform Observability as $h \to 0$

## Lemma (Aydin, Özer, Walterman, IEEE-LCSS'23 )

*Eigenvalues* $\tilde{\mu}_k$ *of* $(\boldsymbol{M}^{-1} + \boldsymbol{J}_h)\boldsymbol{A}_h^2$ *are*

$$\tilde{\mu}_k(h) = \left(\frac{1}{\tilde{\lambda}_k(h)} + \omega_k(h)\right) \lambda_k^2(h), \ \ k = 1, 2, ..., N,$$

*with* $\sqrt{\tilde{\mu}_k(h)} := -\sqrt{\tilde{\mu}_{-k}(h)}$ *for* $k = -1, -2, ..., -N$. *Hence*

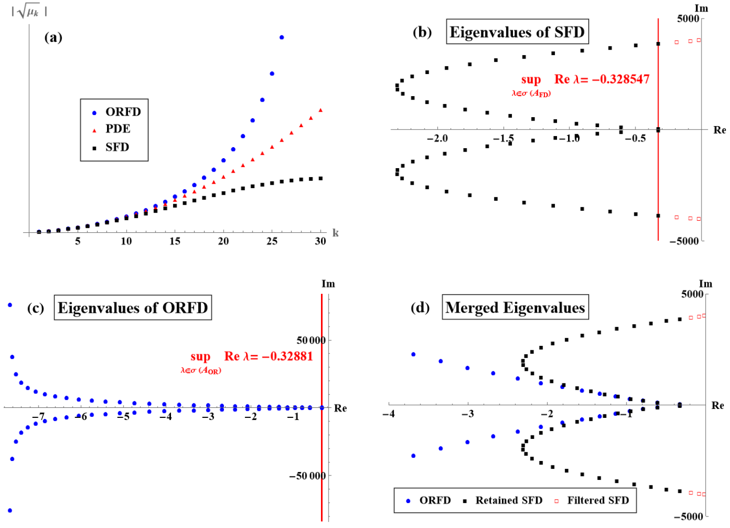
$$\vec{z}(t) = \sum_{k \in K} (a_k e^{i\sqrt{\tilde{\mu}_k(h)} t}) \vec{\phi}_k(h).$$

## Theorem (Aydin, Özer, Walterman, IEEE-LCSS'23 )

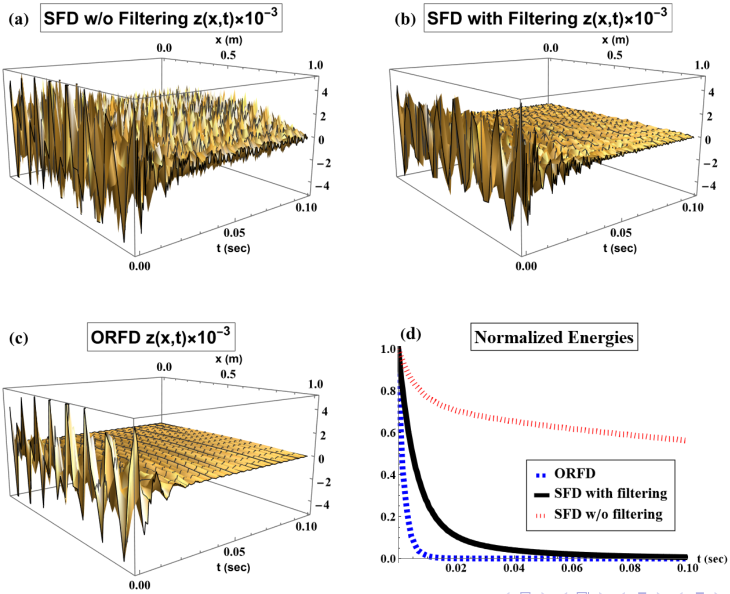
*For any* $T > 0$, *the system* (18) *is observable uniformly as* $h \to 0$:

$$E_h(0) = \frac{L}{2T} \int_0^T \left| \frac{\dot{z}_N}{h} \right|^2 dt.$$

# Eigenvalues of Closed-Loop System



# Energy of Closed-Loop System



# Sensor Failure $\Longrightarrow$ Actuator Design Failure

- $\boxed{\text{Blind model reduction causes uncontrallability.}}$

# Filtering $\Longrightarrow$ Robust Actuator Design

# Order Reduced FD $\Longrightarrow$ Robust Actuator Design

# Ongoing & Future Work I

- Designing a stabilizing controller is an ongoing project. [3]
- The optimal feedback gain and the filtering parameter $\rightarrow$ maximal decay rate? —Ongoing work
- Cantilevered M-M beam: [Özer-Aydin-LCSS'23,ESAIM'23]
- Rao-Nakra beam: [Haider-Özer-in progress].
- Order-reduction for the E-B beam: [Haider-Özer-Aydin'23-preprint].
- Wolfram Demonstration Projects?
  - Three-layer Cantilevered[4] and Hinged[5] demonstrations are **published.**
- The overall analysis here aligns with the multi-layer serially-connected beam designs [Özer,Nicaise,Akil,Regnier'23-arXiv:2303.05882].

[3] A.Ö. Özer, A.K. Aydin, *A novel robust feedback controller design for a cantilevered Mead-Marcus-type sandwich beam model by the order-reduction technique*, in prep.

[4] A.K. Aydin, M. Poynter, A.Ö. Özer, *Feedback Sensor Design for a Cantilevered Three-Layer Sandwich Beam*, **Wolfram Demonstrations Project**, Published: October 6, 2022.

[5] A.K. Aydin, M. Poynter, A.Ö. Özer, *Vibration Suppression on a Hinged Three-Layer Sandwich Beam*, **Wolfram Demonstrations Project,** *Published: January 2, 2023.*